

Decentralized Saddle Point Problems via Non-Euclidean Mirror Prox

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Abstract

We consider smooth convex-concave saddle point problems in the decentralized distributed setting, where a finite-sum objective is distributed among the nodes of a computational network. At each node, the local objective depends on the groups of local and global variables. For such problems, we propose a decentralized distributed algorithm with $O(\varepsilon^{-1})$ communication and oracle calls complexities to achieve accuracy ε in terms of the duality gap and in terms of consensus between nodes. Further, we prove lower bounds for the communication and oracle calls complexities and show that our algorithm matches these bounds, i.e., it is optimal. In contrast to existing decentralized algorithms, our algorithm admits non-euclidean proximal setup, including, e.g., entropic. We illustrate the work of the proposed algorithm on the prominent problem of computing Wasserstein barycenters (WB), where a non-euclidean proximal setup arises naturally in a bilinear saddle point reformulation of the WB problem.

KEYWORDS

Convex optimization, distributed optimization, Wasserstein barycenter, Mirror-Prox

1. Introduction

In the last few years, we observe an increased interest in the research on algorithms for saddle-point problems (SPP), in particular, motivated by modern applications in GANs training [22], reinforcement learning [29, 48, 58], optimal transport [28] and distributed control [43]. In addition to the above modern applications, and besides their classical examples in economics, equilibrium theory, game theory [20], saddle-point problems remain popular in supervised learning (with non-separable loss [30]; with non-separable regularizer [3]), unsupervised learning (discriminative clustering [59]; matrix factorization [4]), image denoising [12, 19], robust optimization [5], optimization with separable or semi-definite constraints [41], and non-smooth optimization via smooth reformulations [45, 47]. Decentralized algorithms for SPPs are also an active area of research [40, 42].

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For convex optimization, the theory of decentralized first-order methods is currently well-developed: the lower bounds on the number of communication rounds and oracle calls are well-known, and algorithms converging according to these lower bounds are developed, see [1, 15, 24, 27, 32, 35, 44, 51, 52, 54, 57]. For convex-concave decentralized SPPs a complete theory exists only in the strongly convex-concave case for unconstrained problems in the euclidean proximal setup [31]. Moreover, the lower bounds from [8, 9, 31] were obtained only for the case when strong convexity and strong concavity parameters are the same.

For non-strongly convex-concave case, distributed SPP with local and global variables were studied in [41], where the authors proposed a subgradient-based algorithm for non-smooth problems with $O(1/\sqrt{N})$ convergence guarantee (N is the number of communication rounds). Paper [61] introduced an Extra-gradient algorithm for distributed multi-block SPP with affine constraints. Their method covers the Euclidean case and the algorithm has $O(1/N)$ convergence rate. Our paper proposes an algorithm based on adding Lagrangian multipliers to consensus constraints, which is analogical to [61], but our method works in a general proximal smooth setup and achieves $O(1/N)$ convergence rate. Moreover, it has an enhanced dependence on the condition number of the network.

An optimal method for convex-concave SPPs is Mirror-prox algorithm [45, 49] with general proximal setup. The Mirror-prox algorithm can be performed in a decentralized manner, however, it is not known whether its optimality is preserved. In this paper, we prove that Mirror-prox remains optimal even in a decentralized case w.r.t. the dependence on the desired accuracy ε and condition number χ of communication network if we split communication and oracle complexities by Chebyshev acceleration trick (see, e.g. [37]).

Finally, we show how the proposed method can be applied to prominent problem of computing Wasserstein barycenters to tackle the problem of instability of regularization-based approaches under a small value of regularizing parameter. The idea is based on the saddle point reformulation of the Wasserstein barycenter problem (see [17]). Wasserstein barycenters, which define the mean of objects that can be modeled as probability measures on a metric space (images, texts, videos), are used in many fields including Bayesian computations [55], texture mixing [50], clustering (k -means for probability measures) [13], shape interpolation and color transferring [53], statistical estimation of template models [10] and neuroimaging [25].

In the numerical experiments conducted on different network architectures, we demonstrate a better approximation of the true barycenter by the proposed distributed Mirror-prox algorithm in comparison with regularization based approaches.

Contribution. Our contribution can be summarized as follows.

- We provide a decentralized Mirror-prox based algorithm for convex-concave SPPs and prove its optimality in the euclidean proximal setup;
- We provide the lower bounds on the number of communication steps and oracle calls per node for convex-concave SPPs in the euclidean proximal setup.

Paper organization. This paper is organized as follows. Section 2 presents a saddle point problem of interest along with its decentralized reformulation. In Section 3, we provide the main algorithm of the paper to solve such kind of problems. In Section 4, we present the lower complexity bounds for saddle point problems without individual variables. Finally in Section 5, we show how the proposed algorithm can be applied to the problem computing Wasserstein barycenters .

Notation. For a prox-function $d(x)$, we define the corresponding Bregman divergence:

$B(x, y) = d(x) - d(y) - \langle \nabla d(y), x - y \rangle$. For some norm $\|\cdot\|$, we define its dual norm $\|\cdot\|_*$ in a usual way: $\|s\|_* = \max_{x \in \mathcal{X}} \{\langle x, s \rangle : \|x\| \leq 1\}$. For two vectors x, y of the same size, denotations x/y and $x \odot y$ stand for the element-wise product and element-wise division respectively. When functions, such as *log* or *exp*, are used on vectors, they are always applied element-wise. We use bold symbol for column vector $\mathbf{x} = [x_1^\top, \dots, x_m^\top]^\top \in \mathbb{R}^{mn}$, where $x_1, \dots, x_m \in \mathbb{R}^n$. Then we refer to the i -th component of vector \mathbf{x} as $x_i \in \mathbb{R}^n$ and to the j -th component of vector x_i as $[x_i]_j$. We denote by $\mathbf{1}$ the vector of ones. For matrices A and B we denote their Kronecker product as $A \otimes B$.

2. Problem statement

In this paper, we consider a saddle point problem of the sum-type with global and local groups of variables

$$\min_{\substack{x \in \bar{\mathcal{X}} \\ \mathbf{p} \in \mathcal{P}}} \max_{\substack{y \in \bar{\mathcal{Y}} \\ \mathbf{q} \in \mathcal{Q}}} f(x, \mathbf{p}, y, \mathbf{q}) \triangleq \frac{1}{m} \sum_{i=1}^m f_i(x, p_i, y, q_i), \quad (1)$$

where $\mathbf{p} \triangleq (p_1^\top \in \mathcal{P}_1 \dots p_m^\top \in \mathcal{P}_m)^\top \in \mathcal{P} \triangleq \mathcal{P}_1 \times \dots \times \mathcal{P}_m$ and $\mathbf{q} \triangleq (q_1^\top \in \mathcal{Q}_1 \dots q_m^\top \in \mathcal{Q}_m)^\top \in \mathcal{Q} \triangleq \mathcal{Q}_1 \times \dots \times \mathcal{Q}_m$. Variables x, p_i, y, q_i have dimensions d_x, d_p, d_y, d_q respectively for all $i = 1, \dots, m$. For problem (1), we make the following assumption.

Assumption 2.1.

- (1) Sets $\mathcal{P}_i, \mathcal{Q}_i$ ($i = 1, \dots, m$) and $\bar{\mathcal{X}}, \bar{\mathcal{Y}}$ are convex and compact.
- (2) Function $f_i(\cdot, \cdot, y, q_i)$ is convex on $\bar{\mathcal{X}} \times \mathcal{P}_i$ for every fixed $y \in \bar{\mathcal{Y}}, q_i \in \mathcal{Q}_i$ for all $i = 1, \dots, m$.
- (3) Function $f_i(x, p_i, \cdot, \cdot)$ is concave on $\bar{\mathcal{Y}} \times \mathcal{Q}_i$ for every fixed $x \in \bar{\mathcal{X}}, p_i \in \mathcal{P}_i$ for all $i = 1, \dots, m$.

2.1. Decentralized setup and communication matrix

We suppose that there is a communication network of agents (machines/computing nodes) which we represent by a connected undirected graph $\mathcal{G} = (V, E)$. Each agent i privately holds its function f_i ($i = 1, \dots, m$). Every pair of agents (i, j) can communicate iff $(i, j) \in E$. We can represent these communication constraints imposed by the network through a particular matrix W satisfying the following assumption

Assumption 2.2.

- (1) W is symmetric positive semi-definite matrix
- (2) (Network compatibility) For all $i, j = 1, \dots, m$ the entry of W : $[W]_{ij} = 0$ if $(i, j) \notin E$ and $i \neq j$.
- (3) (Kernel property) For any $v = [v_1, \dots, v_m]^\top \in \mathbb{R}^m$, $Wv = 0$ if and only if $v_1 = \dots = v_m$, i.e. $\text{Ker } W = \text{span}\{\mathbf{1}\}$.

An example of matrix satisfying this assumption is the graph Laplacian $W \in \mathbb{R}^{m \times m}$:

$$[W]_{ij} \triangleq \begin{cases} -1, & \text{if } (i, j) \in E, \\ \text{deg}(i), & \text{if } i = j, \\ 0, & \text{otherwise,} \end{cases} \quad (2)$$

where $\deg(i)$ is the degree of the node i , i.e., the number of neighbors of the node. The main characteristic of the network is the graph condition number

$$\chi \triangleq \chi(W) \triangleq \lambda_{\max}(W)/\lambda_{\min}^+(W),$$

where $\lambda_{\max}(W)$ and $\lambda_{\min}^+(W)$ denote the maximal and the minimal positive eigenvalues of W respectively.

2.2. Distributed problem reformulation

Let matrices $W_x \in \mathbb{R}^{m \times m}$ and $W_y \in \mathbb{R}^{m \times m}$ be associated with variables x and y , respectively. Introduce communication matrices $\mathbf{W}_x = W_x \otimes I_{d_x}$, $\mathbf{W}_y = W_y \otimes I_{d_y}$. We can equivalently rewrite the problem (1) in a distributed setup replacing the constraint $x_1 = \dots = x_m$ by $\mathbf{W}_x \mathbf{x} = 0$ and $y_1 = \dots = y_m$ by $\mathbf{W}_y \mathbf{y} = 0$, respectively. Let us introduce $F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) \triangleq \sum_{i=1}^m f_i(x_i, p_i, y_i, q_i)$ and $\mathcal{X} = \bar{\mathcal{X}} \times \dots \times \bar{\mathcal{X}}$ and $\mathcal{Y} = \bar{\mathcal{Y}} \times \dots \times \bar{\mathcal{Y}}$.

Assumption 2.3. There exist positive scalars M_x, M_y such that for all $i = 1, \dots, m$ and for any $x_i \in \bar{\mathcal{X}}$, $y_i \in \bar{\mathcal{Y}}$, $p_i \in \mathcal{P}_i$, $q_i \in \mathcal{Q}_i$ it holds $\|\nabla_x f_i(x_i, p_i, y_i, q_i)\|_2 \leq M_x$, $\|\nabla_y f_i(x_i, p_i, y_i, q_i)\|_2 \leq M_y$.

Our method requires rewriting problem (1) by adding Lagrangian multipliers \mathbf{s}, \mathbf{z} . Moreover, we bound the norms of the dual variables at the solution.

Theorem 2.4. *Introduce*

$$\bar{R}_Z^2 = \frac{\bar{\alpha} m M_x^2}{(\lambda_{\min}^+(\mathbf{W}_x))^2}, \quad \bar{R}_S^2 = \frac{\bar{\beta} m M_y^2}{(\lambda_{\min}^+(\mathbf{W}_y))^2},$$

where $\bar{\alpha}, \bar{\beta} \in (1, +\infty) \cup \{+\infty\}$. Problem (1) is equivalent to

$$\min_{\substack{\mathbf{p} \in \mathcal{P}, \mathbf{x} \in \mathcal{X} \\ \|\mathbf{s}\|_2 \leq \bar{R}_S}} \max_{\substack{\mathbf{y} \in \mathcal{Y}, \mathbf{q} \in \mathcal{Q} \\ \|\mathbf{z}\| \leq \bar{R}_Z}} [F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) + \langle \mathbf{z}, \mathbf{W}_x \mathbf{x} \rangle + \langle \mathbf{s}, \mathbf{W}_y \mathbf{y} \rangle]. \quad (3)$$

Bounding the norms of dual variables is a known result in minimization [23, 36]. We prove Theorem 2.4 by considering minimization over $(\mathbf{x}, \mathbf{p}, \mathbf{s})$ and maximization over $(\mathbf{y}, \mathbf{q}, \mathbf{z})$ separately. We decompose a saddle-point problem into two optimization problems and apply to each of them the following result.

Lemma 2.5. *Let $\Theta \subseteq \mathbb{R}^d$ be a closed convex set and $h(\theta) : \Theta \rightarrow \mathbb{R}$ be a convex differentiable function. Consider a problem with affine constraints*

$$\min_{\theta \in \Theta} h(\theta) \quad \text{s.t. } \mathbf{A}\theta = b \quad (4)$$

- (1) Dual problem has a solution ν^* such that $\|\nu^*\|_2^2 \leq \frac{\|\nabla h(\theta^*)\|_2^2}{(\sigma_{\min}^+(\mathbf{A}))^2} =: R_\nu^2$, where $\sigma_{\min}^+(\mathbf{A})$ denotes the minimal non-zero singular value of \mathbf{A} (initially presented in [36]).
- (2) Let $R > R_\nu$ and consider a constrained dual problem

$$\max_{\|\nu\|_2 \leq R} \min_{\theta \in \Theta} [h(\theta) + \langle \nu, \mathbf{A}\theta - b \rangle] = \max_{\|\nu\|_2 \leq R} \varphi(\nu) \quad (5)$$

If $(\tilde{\theta}, \tilde{\nu})$ is a saddle point of (5), then $\tilde{\theta}$ is a solution of (4).

Proof. For part 1, see [36].

Introduce function

$$\psi(\theta) = \max_{\|\nu\|_2 \leq R} [h(\theta) + \langle \nu, \mathbf{A}\theta - b \rangle].$$

Since Θ is a compact set, it holds

$$\max_{\|\nu\|_2 \leq R} \varphi(\nu) = \min_{\theta \in \Theta} \psi(\theta)$$

by Sion-Kakutani theorem (see i.e. Theorem D.4.2 in [6]). Moreover, since

$$\nu^* \in \text{Arg max}_{\nu \in \mathbb{R}^m} \varphi(\nu) \text{ and } \nu^* \in B_R(0),$$

we have

$$\nu^* \in \text{Arg max}_{\|\nu\|_2 \leq R} \varphi(\nu) \text{ and } \max_{\nu \in \mathbb{R}^m} \varphi(\nu) = \max_{\|\nu\|_2 \leq R} \varphi(\nu).$$

Also note that $h(\tilde{\theta}) = \min_{\theta \in \Theta} \psi(\theta)$ by Theorem D.4.1 in [6]. Combining the three facts

$$\tilde{\theta} \in \text{Arg min}_{\theta \in \Theta} \psi(\theta), \nu^* \in \text{Arg max}_{\|\nu\|_2 \leq R} \varphi(\nu), \min_{\theta \in \Theta} \psi(\theta) = \max_{\|\nu\|_2 \leq R} \varphi(\nu)$$

we obtain that $(\tilde{\theta}, \nu^*)$ is a saddle point of (5) by Theorem D.4.1 of [6]. Note that $(\tilde{\theta}, \tilde{\nu})$ is a saddle point of (5), as well, but we do not need $\tilde{\nu}$ in the analysis. Therefore for $\nu \in B_R(0)$ we have

$$\begin{aligned} h(\tilde{\theta}) + \langle \nu^*, \mathbf{A}\tilde{\theta} - b \rangle &\geq h(\tilde{\theta}) + \langle \nu^*, \mathbf{A}\tilde{\theta} - b \rangle \\ \langle \nu^* - \nu, \mathbf{A}\tilde{\theta} - b \rangle &\geq 0. \end{aligned}$$

Taking into account that $\|\nu^*\|_2 \leq R_\nu < R$, we imply $\nu^* \in \text{int}B_R(0)$ and therefore $\mathbf{A}\tilde{\theta} - b = 0$. Therefore, we have

$$h(\hat{\theta}) \leq h(\theta) \quad \forall \theta \in \Theta : \mathbf{A}\theta = b$$

which concludes the proof. \square

Proof of Theorem 2.4. We are free to use Sion-Kakutani theorem since the sets $\mathcal{X}, \mathcal{P}, \mathcal{Y}, \mathcal{Q}$ are compact.

$$\begin{aligned} \min_{\substack{\mathbf{W}_x \mathbf{x}=0 \\ \mathbf{p} \in \mathcal{P}, \mathbf{x} \in \mathcal{X}}} \max_{\substack{\mathbf{W}_y \mathbf{y}=0 \\ \mathbf{y} \in \mathcal{Y}, \mathbf{q} \in \mathcal{Q}}} F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) &= \max_{\substack{\mathbf{W}_y \mathbf{y}=0 \\ \mathbf{y} \in \mathcal{Y}, \mathbf{q} \in \mathcal{Q}}} \min_{\substack{\mathbf{W}_x \mathbf{x}=0 \\ \mathbf{p} \in \mathcal{P}, \mathbf{x} \in \mathcal{X}}} F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) \\ &= \max_{\substack{\mathbf{W}_y \mathbf{y}=0 \\ \mathbf{y} \in \mathcal{Y}, \mathbf{q} \in \mathcal{Q}}} \min_{\mathbf{p} \in \mathcal{P}} \left[\min_{\substack{\mathbf{W}_x \mathbf{x}=0 \\ \mathbf{x} \in \mathcal{X}}} F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) \right] \end{aligned} \quad (6)$$

For any fixed pair $(\mathbf{x}, \mathbf{y}, \mathbf{q}) \in \mathcal{X} \times \mathcal{Z} \times \mathcal{Q}$ function $F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q})$ is convex in \mathbf{p} . Consider problem

$$\min_{\mathbf{x} \in \mathcal{X}} F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) \text{ s.t. } \mathbf{W}_x \mathbf{x} = 0 \quad (7)$$

and denote

$$\mathbf{x}^*(\mathbf{p}, \mathbf{y}, \mathbf{q}) = \arg \min_{\substack{\mathbf{W}_x \mathbf{x} = 0 \\ \mathbf{x} \in \mathcal{X}}} [F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q})].$$

By Lemma 2.5, part 1, there exists a solution of dual to (7) with norm bounded by

$$\frac{\|\nabla_{\mathbf{x}} F(\mathbf{x}^*(\mathbf{p}, \mathbf{y}, \mathbf{q}), \mathbf{p}, \mathbf{y}, \mathbf{q})\|_2^2}{(\lambda_{\min}^+(\mathbf{W}_x))^2} \leq \bar{R}_z. \quad (8)$$

By Lemma 2.5 (part 2), problem (7) is equivalent to

$$\max_{\|\mathbf{z}\|_2 \leq \bar{R}_z} \min_{\mathbf{x} \in \mathcal{X}} [F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) + \langle \mathbf{z}, \mathbf{W}_x \mathbf{x} \rangle] \quad (9)$$

in the sense that for any saddle point $(\hat{\mathbf{x}}(\mathbf{p}, \mathbf{y}, \mathbf{q}), \hat{\mathbf{z}}(\mathbf{x}, \mathbf{y}, \mathbf{q}))$ of (9) we have that $\hat{\mathbf{x}}(\mathbf{p}, \mathbf{y}, \mathbf{q})$ is a solution of (7).

Returning to (6), we get

$$\begin{aligned} & \max_{\substack{\mathbf{W}_y \mathbf{y} = 0 \\ \mathbf{y} \in \mathcal{Y}, \mathbf{q} \in \mathcal{Q}}} \min_{\mathbf{p} \in \mathcal{P}} \left[\min_{\substack{\mathbf{W}_x \mathbf{x} = 0 \\ \mathbf{x} \in \mathcal{X}}} F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) \right] \\ &= \max_{\substack{\mathbf{W}_y \mathbf{y} = 0 \\ \mathbf{y} \in \mathcal{Y}, \mathbf{q} \in \mathcal{Q}}} \min_{\mathbf{p} \in \mathcal{P}} \left[\max_{\|\mathbf{z}\|_2 \leq \bar{R}_z} \min_{\mathbf{x} \in \mathcal{X}} (F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) + \langle \mathbf{z}, \mathbf{W}_x \mathbf{x} \rangle) \right] \\ &= \min_{\mathbf{p} \in \mathcal{P}, \mathbf{x} \in \mathcal{X}} \max_{\substack{\mathbf{y} \in \mathcal{Y}, \mathbf{q} \in \mathcal{Q} \\ \mathbf{W}_y \mathbf{y} = 0, \|\mathbf{z}\|_2 \leq \bar{R}_z}} [F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) + \langle \mathbf{z}, \mathbf{W}_x \mathbf{x} \rangle] \\ &= \min_{\mathbf{p} \in \mathcal{P}, \mathbf{x} \in \mathcal{X}} \max_{\substack{\mathbf{q} \in \mathcal{Q} \\ \|\mathbf{z}\|_2 \leq \bar{R}_z}} \left[\max_{\substack{\mathbf{y} \in \mathcal{Y} \\ \mathbf{W}_y \mathbf{y} = 0}} [F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) + \langle \mathbf{z}, \mathbf{W}_x \mathbf{x} \rangle] \right]. \quad (10) \end{aligned}$$

Now we introduce Lagrange multipliers for the constraints $\mathbf{W}_y \mathbf{y} = 0$, as well. Analogously to the case with $\mathbf{W}_x \mathbf{x} = 0$ constraints, consider a problem

$$\max_{\mathbf{y} \in \mathcal{Y}} [F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) + \langle \mathbf{z}, \mathbf{W}_x \mathbf{x} \rangle] \text{ s.t. } \mathbf{W}_y \mathbf{y} = 0 \quad (11)$$

and introduce its solution

$$\mathbf{y}^*(\mathbf{x}, \mathbf{p}, \mathbf{q}, \mathbf{z}) = \arg \max_{\substack{\mathbf{W}_y \mathbf{y} = 0 \\ \mathbf{y} \in \mathcal{Y}}} [F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) + \langle \mathbf{z}, \mathbf{W}_x \mathbf{x} \rangle].$$

Analogously, the dual solution norm to problem (11) can be bounded as

$$\frac{\|\nabla_{\mathbf{y}}F(\mathbf{x}, \mathbf{p}, \mathbf{y}^*(\mathbf{x}, \mathbf{p}, \mathbf{q}, \mathbf{z}), \mathbf{q})\|_2^2}{(\lambda_{\min}^+(\mathbf{W}_{\mathbf{y}}))^2} \leq \bar{R}_{\mathcal{S}}^2, \beta \in (1, +\infty]. \quad (12)$$

And we get a saddle-point reformulation of (11):

$$\min_{\|\mathbf{s}\|_2 \leq \bar{R}_{\mathcal{S}}} \max_{\mathbf{q} \in \mathcal{Q}} [F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) + \langle \mathbf{z}, \mathbf{W}_{\mathbf{x}}\mathbf{x} \rangle + \langle \mathbf{s}, \mathbf{W}_{\mathbf{y}}\mathbf{y} \rangle].$$

Substituting this reformulation into (10), we obtain

$$\begin{aligned} & \min_{\mathbf{p} \in \mathcal{P}, \mathbf{x} \in \mathcal{X}} \max_{\substack{\mathbf{q} \in \mathcal{Q} \\ \|\mathbf{z}\|_2 \leq \bar{R}_{\mathcal{Z}}}} \left[\max_{\substack{\mathbf{y} \in \mathcal{Y} \\ \mathbf{W}_{\mathbf{y}}\mathbf{y}=0}} [F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) + \langle \mathbf{z}, \mathbf{W}_{\mathbf{x}}\mathbf{x} \rangle] \right] \\ &= \min_{\mathbf{p} \in \mathcal{P}, \mathbf{x} \in \mathcal{X}} \max_{\substack{\mathbf{q} \in \mathcal{Q} \\ \|\mathbf{z}\|_2 \leq \bar{R}_{\mathcal{Z}}}} \left[\min_{\|\mathbf{s}\|_2 \leq \bar{R}_{\mathcal{S}}} \max_{\mathbf{y} \in \mathcal{Y}} [F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) + \langle \mathbf{z}, \mathbf{W}_{\mathbf{x}}\mathbf{x} \rangle + \langle \mathbf{s}, \mathbf{W}_{\mathbf{y}}\mathbf{y} \rangle] \right] \\ &= \min_{\mathbf{p} \in \mathcal{P}, \mathbf{x} \in \mathcal{X}} \max_{\substack{\mathbf{y} \in \mathcal{Y}, \mathbf{q} \in \mathcal{Q} \\ \|\mathbf{z}\|_2 \leq \bar{R}_{\mathcal{Z}}}} \min_{\|\mathbf{s}\|_2 \leq \bar{R}_{\mathcal{S}}} [F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) + \langle \mathbf{z}, \mathbf{W}_{\mathbf{x}}\mathbf{x} \rangle + \langle \mathbf{s}, \mathbf{W}_{\mathbf{y}}\mathbf{y} \rangle] \\ &= \max_{\substack{\mathbf{y} \in \mathcal{Y}, \mathbf{q} \in \mathcal{Q} \\ \|\mathbf{z}\|_2 \leq \bar{R}_{\mathcal{Z}}}} \min_{\substack{\mathbf{p} \in \mathcal{P}, \mathbf{x} \in \mathcal{X} \\ \|\mathbf{s}\|_2 \leq \bar{R}_{\mathcal{S}}}} [F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) + \langle \mathbf{z}, \mathbf{W}_{\mathbf{x}}\mathbf{x} \rangle + \langle \mathbf{s}, \mathbf{W}_{\mathbf{y}}\mathbf{y} \rangle], \end{aligned} \quad (13)$$

which is an equivalent reformulation of (1). Note that cases $\bar{R}_{\mathcal{S}} = +\infty, \bar{R}_{\mathcal{Z}} = +\infty$ are also supported in this proof. A min-max reformulation is obtained analogously by adding Lagrange multipliers in different order. \square

From Theorem 2.4 we immediately obtain the following corollary by setting $\bar{\alpha} = \bar{\beta} = +\infty$.

Corollary 2.6. *Problem (1) can be equivalently rewritten as follows*

$$\min_{\substack{\mathbf{x} \in \mathcal{X}, \mathbf{p} \in \mathcal{P} \\ \mathbf{s} \in \mathbb{R}^{m_{\mathbf{d}_y}}}} \max_{\substack{\mathbf{y} \in \mathcal{Y}, \mathbf{q} \in \mathcal{Q} \\ \mathbf{z} \in \mathbb{R}^{m_{\mathbf{d}_x}}}} [F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) + \langle \mathbf{z}, \mathbf{W}_{\mathbf{x}}\mathbf{x} \rangle + \langle \mathbf{s}, \mathbf{W}_{\mathbf{y}}\mathbf{y} \rangle], \quad (14)$$

in the sense that for any saddle point $(\mathbf{x}^, \mathbf{p}^*, \mathbf{y}^*, \mathbf{q}^*, \mathbf{s}^*, \mathbf{z}^*)$ of (14) we have $x_1^* = \dots = x_m^* = x^*$, $y_1^* = \dots = y_m^* = y^*$, and the point $(x^*, \mathbf{p}^*, y^*, \mathbf{q}^*)$ is a saddle point of (1).*

3. Distributed algorithm for saddle-point problems

Now we provide a decentralized distributed algorithm to solve saddle-point problem (1). The pseudo-code is listed in Algorithm 1 and the main result on convergence rate is given in Theorem 3.5. The main idea is to use reformulation (14) and apply mirror prox algorithm [45] for its solution. This requires careful analysis in two aspects. First, the Lagrange multipliers \mathbf{z}, \mathbf{s} are not constrained, while the convergence rate result for the classical Mirror-Prox algorithm [45] is proved for problems on compact sets. Second, we need to show that the updates can be organized via only local communications between the nodes in the network.

3.1. Algorithm

For each variable $t_i \in \{x_i, p_i, y_i, q_i, s_i, z_i\}$ and corresponding set $\mathcal{T}_i \in \{\bar{\mathcal{X}}, \mathcal{P}_i, \bar{\mathcal{Y}}, \mathcal{Q}_i, \mathbb{R}^{d_y}, \mathbb{R}^{d_x}\}$, we assume there is a norm $\|t_i\|_{\mathcal{T}_i}$. Our analysis supports arbitrary norms for x_i, p_i, y_i, q_i , but we use the Euclidean norm for dual variables s_i, z_i , i.e. $\|s_i\|_{\mathcal{S}_i} = \|s_i\|_2$, $\|z_i\|_{\mathcal{Z}_i} = \|z_i\|_2$. With each norm $\|t_i\|_{\mathcal{T}_i}$ we associate a 1-strongly convex w.r.t. this norm prox-function $d_{\mathcal{T}_i}(t_i)$ and the corresponding Bregman divergence $B_{\mathcal{T}_i}(t_i, \check{t}_i) = d_{\mathcal{T}_i}(t_i) - d_{\mathcal{T}_i}(\check{t}_i) - \langle \nabla d_{\mathcal{T}_i}(\check{t}_i), t_i - \check{t}_i \rangle$. For dual variables we have prox-functions $d_{\mathcal{S}_i}(u_i) = \frac{1}{2} \|u_i\|_2^2$, $d_{\mathcal{Z}_i}(z_i) = \frac{1}{2} \|z_i\|_2^2$ and Bregman divergences $B_{\mathcal{S}_i}(s_i, \check{s}_i) = \frac{1}{2} \|s_i - \check{s}_i\|_2^2$, $B_{\mathcal{Z}_i}(z_i, \check{z}_i) = \frac{1}{2} \|z_i - \check{z}_i\|_2^2$.

Having introduced $B_{\mathcal{T}_i}(t_i, \check{t}_i)$, we define the following *Mirror step*:

$$\text{Mirr}(g_i; t_i; \mathcal{T}_i) = \arg \min_{t \in \mathcal{T}_i} [\langle g_i, t \rangle + B_{\mathcal{T}_i}(t, t_i)], \quad (15)$$

where g_i is an element of the corresponding dual space which defines the step direction.

Our decentralized algorithm for saddle-point problem (14) is listed as Algorithm 1. In each iteration each agent i makes two Mirror step updates in each of its six local variables. Besides using respective gradient of the local objective f_i to define the step direction, some updates include aggregating the variables of other agents. Consider for example the update $x_i^{k+\frac{1}{2}}$, in which agent i needs to calculate the sum $\sum_j [\mathbf{W}_x]_{ij} z_j^k$. Due to the Assumption 2.2, i.e. network compatibility of matrix W , this update requires aggregating local only from neighbouring agents. Thus, each iteration of the algorithm is performed in a distributed manner.

In Algorithm 1, we denote $\nabla_t f_i^\ell = \nabla_t f_i(x_i^\ell, p_i^\ell, y_i^\ell, q_i^\ell)$ for $t \in \{x, p, y, q\}$ and $\ell \in \{k, k + \frac{1}{2}\}$

In the next two subsections, we introduce two necessary components which allow us to prove the convergence rate theorem for our algorithm. These are smoothness assumptions and localization of the solution to the saddle-point problem (14).

3.2. Localizing the solution and smoothness assumptions

As it was noted above, the standard analysis of Mirror-Prox requires the feasible sets to be compact. Although we run Mirror-Prox algorithm on problem (14) with unconstrained variables \mathbf{s} and \mathbf{z} , we still can bound these variables according to Theorem 2.4.

Lemma 3.1. *Let Assumption 2.3 be satisfied and $R_{\mathcal{Z}}^2 = 2mM_x^2(\lambda_{\min}^+(\mathbf{W}_x))^{-2}$, $R_{\mathcal{S}}^2 = 2mM_y^2(\lambda_{\min}^+(\mathbf{W}_y))^{-2}$, where $\lambda_{\min}^+(\cdot)$ is the minimal non-zero eigenvalue of matrix. Then there exists a saddle point $(\mathbf{x}^*, \mathbf{p}^*, \mathbf{y}^*, \mathbf{q}^*, \mathbf{s}^*, \mathbf{z}^*)$ of problem (14) such that $\|\mathbf{s}^*\|_2 \leq R_{\mathcal{S}}$, $\|\mathbf{z}^*\|_2 \leq R_{\mathcal{Z}}$.*

Proof. Putting $\bar{\alpha} = \bar{\beta} = 2$ in Theorem 2.4, we immediately obtain the proof. \square

Next, we introduce the second important component of the convergence rate analysis, namely the smoothness assumption on the objective F . To set the stage we first introduce a general definition of Lipschitz-smooth function of two variables. Having defined norms and Bregman divergences, let $t_i \in \{x_i, p_i, s_i, y_i, q_i, z_i\}$ and correspondingly $\mathcal{T}_i \in \{\bar{\mathcal{X}}, \mathcal{P}_i, \mathbb{R}^{d_y}, \bar{\mathcal{Y}}, \mathcal{Q}_i, \mathbb{R}^{d_x}\}$. Introduce $\mathbf{t} = (t_1^\top, \dots, t_m^\top)^\top$ and $\mathcal{T} = \mathcal{T}_1 \times \dots \times \mathcal{T}_m$ we define $\|\mathbf{t}\|_{\mathcal{T}}^2 = \sum_{i=1}^m \|t_i\|_{\mathcal{T}_i}^2$.

Algorithm 1 Decentralized Mirror-Prox

Require: Initial guess $(\mathbf{x}^0, \mathbf{p}^0, \mathbf{y}^0, \mathbf{q}^0, \mathbf{z}^0, \mathbf{s}^0)$ such that $\mathbf{t} = [t_1^\top, \dots, t_m^\top]^\top$, $t_i = \arg \min_{t \in \mathcal{T}_i} d_{\mathcal{T}_i}(t)$ for each $t_i \in \{x_i, p_i, y_i, q_i, z_i, s_i\}$ and corresponding set $\mathcal{T}_i \in$

$$\{\mathcal{X}, \mathcal{P}_i, \mathcal{Y}, \mathcal{Q}_i, \mathbb{R}^{d_y}, \mathbb{R}^{d_x}\}, i = 1, \dots, m.$$

- 1: **for** $k = 0, 1, \dots, N - 1$ **do**
- 2: Compute $\tilde{\mathbf{x}}^k = \mathbf{W}_x \mathbf{x}^k$; $\tilde{\mathbf{s}}^k = \mathbf{W}_y \mathbf{s}^k$; $\tilde{\mathbf{y}}^k = \mathbf{W}_y \mathbf{y}^k$; $\tilde{\mathbf{z}}^k = \mathbf{W}_x \mathbf{z}^k$.
- 3: Each node computes

$$\begin{aligned} x_i^{k+\frac{1}{2}} &= \text{Mirr} \left(\alpha (\nabla_x f_i^k + \tilde{z}_i^k); x_i^k; \bar{\mathcal{X}} \right), & s_i^{k+\frac{1}{2}} &= \text{Mirr} \left(\alpha \tilde{y}_i^k; s_i^k; \mathbb{R}^{d_y} \right), \\ y_i^{k+\frac{1}{2}} &= \text{Mirr} \left(-\alpha (\nabla_y f_i^k + \tilde{s}_i^k); y_i^k; \bar{\mathcal{Y}} \right), & z_i^{k+\frac{1}{2}} &= \text{Mirr} \left(-\alpha \tilde{x}_i^k; z_i^k; \mathbb{R}^{d_x} \right). \end{aligned}$$

- 4: Compute $\tilde{\mathbf{x}}^{k+\frac{1}{2}} = \mathbf{W}_x \mathbf{x}^{k+\frac{1}{2}}$; $\tilde{\mathbf{s}}^{k+\frac{1}{2}} = \mathbf{W}_y \mathbf{s}^{k+\frac{1}{2}}$; $\tilde{\mathbf{y}}^{k+\frac{1}{2}} = \mathbf{W}_y \mathbf{y}^{k+\frac{1}{2}}$; $\tilde{\mathbf{z}}^{k+\frac{1}{2}} = \mathbf{W}_x \mathbf{z}^{k+\frac{1}{2}}$.
- 5: Each node computes

$$\begin{aligned} x_i^{k+1} &= \text{Mirr} \left(\alpha (\nabla_p f_i^{k+\frac{1}{2}} + \tilde{z}_i^{k+\frac{1}{2}}); x_i^k; \bar{\mathcal{X}} \right), & s_i^{k+1} &= \text{Mirr} \left(\alpha \tilde{y}_i^{k+\frac{1}{2}}; s_i^k; \mathbb{R}^{d_y} \right), \\ y_i^{k+1} &= \text{Mirr} \left(-\alpha (\nabla_r f_i^{k+\frac{1}{2}} + \tilde{s}_i^{k+\frac{1}{2}}); y_i^k; \bar{\mathcal{Y}} \right), & z_i^{k+1} &= \text{Mirr} \left(-\alpha \tilde{x}_i^{k+\frac{1}{2}}; z_i^k; \mathbb{R}^{d_x} \right). \end{aligned}$$

6: **end for**

Ensure: For $\mathbf{t} \in \{\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}, \mathbf{s}, \mathbf{z}\}$ compute $\hat{\mathbf{t}}^N = \frac{1}{N} \sum_{k=0}^{N-1} \mathbf{t}^{k+\frac{1}{2}}$.

Definition 3.2. Consider norms $\|\cdot\|_\xi$, $\|\cdot\|_\eta$ and their dual norms $\|\cdot\|_{\xi,*}$, $\|\cdot\|_{\eta,*}$. A differentiable function $G(\xi, \eta)$ is called $(L_{\xi\xi}, L_{\xi\eta}, L_{\eta\xi}, L_{\eta\eta})$ -smooth w.r.t. norms $\|\cdot\|_\xi$, $\|\cdot\|_\eta$ if

$$\begin{aligned} \|\nabla_\xi G(\xi, \eta) - \nabla_\xi G(\xi', \eta)\|_{\xi,*} &\leq L_{\xi\xi} \|\xi - \xi'\|_\xi, \\ \|\nabla_\xi G(\xi, \eta) - \nabla_\xi G(\xi, \eta')\|_{\xi,*} &\leq L_{\xi\eta} \|\eta - \eta'\|_\eta, \\ \|\nabla_\eta G(\xi, \eta) - \nabla_\eta G(\xi', \eta)\|_{\eta,*} &\leq L_{\eta\xi} \|\xi - \xi'\|_\xi, \\ \|\nabla_\eta G(\xi, \eta) - \nabla_\eta G(\xi, \eta')\|_{\eta,*} &\leq L_{\eta\eta} \|\eta - \eta'\|_\eta. \end{aligned}$$

In order to formulate the smoothness assumptions on function F , we group the minimization variables (\mathbf{x}, \mathbf{p}) and maximization variables (\mathbf{y}, \mathbf{q}) .

Assumption 3.3. The function F is $(L_{(\mathbf{x}, \mathbf{p})(\mathbf{x}, \mathbf{p})}, L_{(\mathbf{x}, \mathbf{p})(\mathbf{y}, \mathbf{q})}, L_{(\mathbf{y}, \mathbf{q})(\mathbf{x}, \mathbf{p})}, L_{(\mathbf{y}, \mathbf{q})(\mathbf{y}, \mathbf{q})})$ -smooth w.r.t. norms $\|\cdot\|_{(\mathcal{X}, \mathcal{P})}$, $\|\cdot\|_{(\mathcal{Y}, \mathcal{Q})}$ defined as

$$\|(\mathbf{x}, \mathbf{p})\|_{(\mathcal{X}, \mathcal{P})}^2 = \|\mathbf{x}\|_{\mathcal{X}}^2 + \|\mathbf{p}\|_{\mathcal{P}}^2, \quad \|(\mathbf{y}, \mathbf{q})\|_{(\mathcal{Y}, \mathcal{Q})}^2 = \|\mathbf{y}\|_{\mathcal{Y}}^2 + \|\mathbf{q}\|_{\mathcal{Q}}^2.$$

3.3. Main result

To present a convergence bound for Algorithm 1, we introduce additional notation. For each $\mathcal{T}_i \in \{\bar{\mathcal{X}}, \mathcal{P}_i, \bar{\mathcal{Y}}, \mathcal{Q}_i, \mathbb{R}^{d_y}, \mathbb{R}^{d_x}\}$, we introduce

$$R_{\mathcal{T}_i}^2 = \max_{t, t' \in \mathcal{T}_i} d_{\mathcal{T}_i}(t) - d_{\mathcal{T}_i}(t'). \quad (16)$$

For simplicity of further derivations, we aggregate all the variables in two blocks: minimization variables and maximization variables. Introducing $\mathbf{u} = (\mathbf{x}^\top, \mathbf{p}^\top, \mathbf{s}^\top)^\top$ and $\mathbf{v} = (\mathbf{y}^\top, \mathbf{q}^\top, \mathbf{z}^\top)^\top$ makes it possible to rewrite the problem (14) in the following simplified form.

$$\min_{\mathbf{u} \in \mathcal{A}_U} \max_{\mathbf{v} \in \mathcal{A}_V} S(\mathbf{u}, \mathbf{v}) \triangleq F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) + \langle \mathbf{s}, \mathbf{W}_y \mathbf{y} \rangle + \langle \mathbf{z}, \mathbf{W}_x \mathbf{x} \rangle, \quad (17)$$

where $\mathcal{A}_U = \mathcal{X} \times \mathcal{P} \times \mathbb{R}^{nd_a}$ and $\mathcal{A}_V = \mathcal{Y} \times \mathcal{Q} \times \mathbb{R}^{nd_p}$. The sets \mathcal{A}_U and \mathcal{A}_V are closed and convex but unbounded. We also define $\mathcal{U} = \mathcal{X} \times \mathcal{P} \times \{\mathbf{s} : \|\mathbf{s}\|_2 \leq R_S\}$ and $\mathcal{V} = \mathcal{Y} \times \mathcal{Q} \times \{\mathbf{z} : \|\mathbf{z}\|_2 \leq R_Z\}$. Convexity properties of F imply that $S(\cdot, \mathbf{v})$ is convex for any fixed \mathbf{v} and $S(\mathbf{u}, \cdot)$ is concave for any fixed \mathbf{u} .

Introduce norms

$$\|\mathbf{u}\|_{\mathcal{U}}^2 = \frac{\|\mathbf{x}\|_{\mathcal{X}}^2}{R_{\mathcal{X}}^2} + \frac{\|\mathbf{p}\|_{\mathcal{P}}^2}{R_{\mathcal{P}}^2} + \frac{\|\mathbf{s}\|_2^2}{R_S^2}, \quad \|\mathbf{v}\|_{\mathcal{V}}^2 = \frac{\|\mathbf{y}\|_{\mathcal{Y}}^2}{R_{\mathcal{Y}}^2} + \frac{\|\mathbf{q}\|_{\mathcal{Q}}^2}{R_{\mathcal{Q}}^2} + \frac{\|\mathbf{z}\|_2^2}{R_Z^2}. \quad (18)$$

Also introduce prox-functions $d_U(\mathbf{u})$, $d_V(\mathbf{v})$ and Bregman divergences

$$B_U(\mathbf{u}, \check{\mathbf{u}}) = \frac{B_{\mathcal{X}}(\mathbf{x}, \check{\mathbf{x}})}{R_{\mathcal{X}}^2} + \frac{B_{\mathcal{P}}(\mathbf{p}, \check{\mathbf{p}})}{R_{\mathcal{P}}^2} + \frac{B_S(\mathbf{s}, \check{\mathbf{s}})}{R_S^2},$$

$$B_V(\mathbf{v}, \check{\mathbf{v}}) = \frac{B_{\mathcal{Y}}(\mathbf{y}, \check{\mathbf{y}})}{R_{\mathcal{Y}}^2} + \frac{B_{\mathcal{Q}}(\mathbf{q}, \check{\mathbf{q}})}{R_{\mathcal{Q}}^2} + \frac{B_Z(\mathbf{z}, \check{\mathbf{z}})}{R_Z^2}.$$

Lemma 3.4. *Function $S(\mathbf{u}, \mathbf{v})$ defined in (17) is $(L_{\mathbf{u}\mathbf{u}}, L_{\mathbf{u}\mathbf{v}}, L_{\mathbf{v}\mathbf{u}}, L_{\mathbf{v}\mathbf{v}})$ -smooth, where*

$$L_{\mathbf{u}\mathbf{u}} = L_{(\mathbf{x}, \mathbf{p})(\mathbf{x}, \mathbf{p})}(R_{\mathcal{X}}^2 + R_{\mathcal{P}}^2), \quad L_{\mathbf{v}\mathbf{v}} = L_{(\mathbf{y}, \mathbf{q})(\mathbf{y}, \mathbf{q})}(R_{\mathcal{Y}}^2 + R_{\mathcal{Q}}^2),$$

$$L_{\mathbf{u}\mathbf{v}} = \sqrt{2} \left(L_{(\mathbf{x}, \mathbf{p})(\mathbf{y}, \mathbf{q})} \sqrt{(R_{\mathcal{X}}^2 + R_{\mathcal{P}}^2)(R_{\mathcal{Y}}^2 + R_{\mathcal{Q}}^2)} + R_{\mathcal{X}} R_Z \|\mathbf{W}_x\|_{2 \rightarrow (\mathbf{x}, *)} + R_{\mathcal{Y}} R_S \|\mathbf{W}_y\|_{\mathbf{y} \rightarrow 2} \right),$$

$$L_{\mathbf{v}\mathbf{u}} = \sqrt{2} \left(L_{(\mathbf{y}, \mathbf{q})(\mathbf{x}, \mathbf{p})} \sqrt{(R_{\mathcal{X}}^2 + R_{\mathcal{P}}^2)(R_{\mathcal{Y}}^2 + R_{\mathcal{Q}}^2)} + R_{\mathcal{X}} R_Z \|\mathbf{W}_x\|_{\mathbf{x} \rightarrow 2} + R_{\mathcal{Y}} R_S \|\mathbf{W}_y\|_{2 \rightarrow (\mathbf{y}, *)} \right).$$

The proof is mostly technical and therefore omitted to the Appendix 8.

Let us now discuss the second main aspect of the analysis, i.e. unboundedness of the feasible sets \mathcal{A}_U , \mathcal{A}_V , which does not allow to directly apply the standard analysis of [45]. Due to the special structure of the problem (14) and the Assumption 2.3, we have at the disposal bounds R_Z, R_S defined in Lemma 3.1 such that the optimal values \mathbf{z}^* and \mathbf{s}^* satisfy $\|\mathbf{z}^*\|_2 \leq R_Z$, $\|\mathbf{s}^*\|_2 \leq R_S$. We have that the saddle-point $(\mathbf{u}^*, \mathbf{v}^*)$ belongs to $\mathcal{U} \times \mathcal{V}$. Let us define

$$L_{\zeta} = 2 \max\{L_{\mathbf{u}\mathbf{u}}, L_{\mathbf{u}\mathbf{v}}, L_{\mathbf{v}\mathbf{u}}, L_{\mathbf{v}\mathbf{v}}\}. \quad (19)$$

Finally, applying the analysis of Mirror-Prox, we obtain the main result.

Theorem 3.5. *Let Assumptions 2.2, 2.3 and 3.3 hold. Let Algorithm 1 be run for N iterations with stepsize $\alpha = 1/L_\zeta$, where L_ζ is defined in (19). Introduce $\widehat{\mathbf{t}}^N = \frac{1}{m}\mathbf{1}\mathbf{1}^\top \widehat{\mathbf{t}}^N$ for $\mathbf{t} \in \{\mathbf{x}, \mathbf{y}, \mathbf{p}, \mathbf{q}\}$. We have*

$$\max_{\substack{\mathbf{y} \in \mathcal{Y}, \mathbf{q} \in \mathcal{Q} \\ \mathbf{W}_y \mathbf{y} = 0}} F(\widehat{\mathbf{x}}^N, \widehat{\mathbf{p}}^N, \mathbf{y}, \mathbf{q}) - \min_{\substack{\mathbf{x} \in \mathcal{X}, \mathbf{p} \in \mathcal{P} \\ \mathbf{W}_x \mathbf{x} = 0}} F(\mathbf{x}, \mathbf{p}, \widehat{\mathbf{y}}^N, \widehat{\mathbf{q}}^N) \leq \frac{(4 + 17\sqrt{2})L_\zeta}{N}$$

and

$$\|\mathbf{W}_x \widehat{\mathbf{x}}^N\|_2 \leq \frac{17L_\zeta}{NR_Z}, \quad \|\mathbf{W}_y \widehat{\mathbf{y}}^N\|_2 \leq \frac{17L_\zeta}{NR_S}.$$

Constant L_ζ is dependent on problem parameters as defined in (19). Let us provide an explicit dependence for completeness.

Corollary 3.6. *Under conditions of Theorem 3.5 we have*

$$\begin{aligned} & \max_{\substack{\mathbf{y} \in \mathcal{Y}, \mathbf{q} \in \mathcal{Q} \\ \mathbf{W}_y \mathbf{y} = 0}} F(\widehat{\mathbf{x}}^N, \widehat{\mathbf{p}}^N, \mathbf{y}, \mathbf{q}) - \min_{\substack{\mathbf{x} \in \mathcal{X}, \mathbf{p} \in \mathcal{P} \\ \mathbf{W}_x \mathbf{x} = 0}} F(\mathbf{x}, \mathbf{p}, \widehat{\mathbf{y}}^N, \widehat{\mathbf{q}}^N) \\ & \leq \frac{(72 + 8\sqrt{2})m}{N} \left(L_{(\mathbf{x}, \mathbf{p})(\mathbf{x}, \mathbf{p})}(R_{\mathcal{X}}^2 + R_{\mathcal{P}}^2) + L_{(\mathbf{y}, \mathbf{q})(\mathbf{y}, \mathbf{q})}(R_{\mathcal{Y}}^2 + R_{\mathcal{Q}}^2) \right. \\ & \quad \left. + (L_{(\mathbf{x}, \mathbf{p})(\mathbf{y}, \mathbf{q})} + L_{(\mathbf{y}, \mathbf{q})(\mathbf{x}, \mathbf{p})}) \sqrt{(R_{\mathcal{X}}^2 + R_{\mathcal{P}}^2)(R_{\mathcal{Y}}^2 + R_{\mathcal{Q}}^2)} \right. \\ & \quad \left. + \frac{M_x R_{\mathcal{X}} (\|\mathbf{W}_x\|_{2 \rightarrow (\mathbf{x}, *)} + \|\mathbf{W}_x\|_{\mathbf{x} \rightarrow 2})}{\lambda_{\min}^+(\mathbf{W}_x)} + \frac{M_y R_{\mathcal{Y}} (\|\mathbf{W}_y\|_{2 \rightarrow (\mathbf{y}, *)} + \|\mathbf{W}_y\|_{\mathbf{y} \rightarrow 2})}{\lambda_{\min}^+(\mathbf{W}_y)} \right). \end{aligned}$$

To prove Theorem 3.5 we first show that the iterates of Algorithm 1 naturally correspond to the iterates of a general Mirror-Prox algorithm applied to problem (14). Then we extend the standard analysis of the general Mirror-Prox algorithm to account for unbounded feasible sets.

The feasible set $\mathcal{A}_U \times \mathcal{A}_V$ is separable w.r.t. each local feasible set $\bar{\mathcal{X}} \times \mathcal{P}_i \times \mathbb{R}^{d_q} \times \bar{\mathcal{Y}} \times \bar{\mathcal{Q}}_i \times \mathbb{R}^{d_p}$ for an agent i . Thus, if we define a variable $\zeta = (\mathbf{u}^\top, \mathbf{v}^\top)^\top \in \mathcal{A} := \mathcal{A}_U \times \mathcal{A}_V$, and the operator $g(\zeta)$

$$g(\zeta) = \begin{bmatrix} \nabla_{\mathbf{u}} S(\mathbf{u}, \mathbf{v}) \\ -\nabla_{\mathbf{v}} S(\mathbf{u}, \mathbf{v}) \end{bmatrix} = \begin{bmatrix} \nabla_{\mathbf{x}} F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) + \mathbf{W}_x \mathbf{z} \\ \nabla_{\mathbf{p}} F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) \\ \mathbf{W}_y \mathbf{y} \\ -\nabla_{\mathbf{y}} F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) - \mathbf{W}_y \mathbf{s} \\ -\nabla_{\mathbf{q}} F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) \\ -\mathbf{W}_x \mathbf{x} \end{bmatrix},$$

then the updates of Algorithm 1 are equivalent to the updates of a general Mirror-Prox algorithm listed as Algorithm 2.

Algorithm 2 Mirror-Prox

Require: Starting point $\zeta^0 = \arg \min_{\zeta \in Q} d_\zeta(\zeta)$, stepsize $\alpha > 0$.

1: **for** $k = 0, 1, \dots$ **do**
 2:

$$\zeta^{k+\frac{1}{2}} = \arg \min_{\zeta \in Q} \alpha \langle g(\zeta^k), \zeta \rangle + B(\zeta, \zeta^k) \quad (20)$$

$$\zeta^{k+1} = \arg \min_{\zeta \in Q} \alpha \langle g(\zeta^{k+\frac{1}{2}}), \zeta \rangle + B(\zeta, \zeta^k) \quad (21)$$

3: **end for**

Ensure: $\widehat{\zeta}^k = \frac{1}{k} \sum_{\ell=0}^{k-1} \zeta^{\ell+\frac{1}{2}}$.

Next, we analyze the general Mirror-Prox Algorithm 2. Let us define the norm $\|\zeta\|_\zeta^2 = \|\mathbf{u}\|_{\mathcal{U}}^2 + \|\mathbf{v}\|_{\mathcal{V}}^2$, the corresponding prox-function $d_\zeta(\zeta)$ and Bregman divergence $B_\zeta(\zeta, \check{\zeta}) = B_{\mathbf{u}}(\mathbf{u}, \check{\mathbf{u}}) + B_{\mathbf{v}}(\mathbf{v}, \check{\mathbf{v}})$. Note that the corresponding dual norm is $\|\zeta\|_{\zeta,*}^2 = \|\mathbf{u}\|_{\mathcal{U},*}^2 + \|\mathbf{v}\|_{\mathcal{V},*}^2$. Under these definitions, following the standard analysis in [11], we obtain that the operator $g(\zeta)$ is L_ζ -Lipschitz-continuous with respect to the norm $\|\zeta\|_\zeta$ with

$$L_\zeta = 2 \max\{L_{\mathbf{u}\mathbf{u}}, L_{\mathbf{v}\mathbf{v}}, L_{\mathbf{u}\mathbf{v}}, L_{\mathbf{v}\mathbf{u}}\}. \quad (22)$$

Next, we analyze the iterations of Algorithm 2 under the assumption that the operator $g(\zeta)$ is L_ζ -Lipschitz-continuous. Let us fix some iteration $k \geq 0$. By the optimality conditions in (20), we have, for any $\zeta \in Q$,

$$\alpha \langle g(\zeta^k) + \nabla d_\zeta(\zeta^{k+\frac{1}{2}}) - \nabla d_\zeta(\zeta^k), \zeta - \zeta^{k+\frac{1}{2}} \rangle \geq 0, \quad (23)$$

$$\alpha \langle g(\zeta^{k+\frac{1}{2}}) + \nabla d_\zeta(\zeta^{k+1}) - \nabla d_\zeta(\zeta^k), \zeta - \zeta^{k+1} \rangle \geq 0. \quad (24)$$

Whence, for all $\zeta \in Q$,

$$\begin{aligned} \langle g(\zeta^{k+\frac{1}{2}}), \zeta^{k+\frac{1}{2}} - \zeta \rangle &= \langle g(\zeta^{k+\frac{1}{2}}), \zeta^{k+1} - \zeta \rangle + \langle g(\zeta^{k+\frac{1}{2}}), \zeta^{k+\frac{1}{2}} - \zeta^{k+1} \rangle \\ &\stackrel{(24)}{\leq} \frac{1}{\alpha} \langle \nabla d_\zeta(\zeta^k) - \nabla d_\zeta(\zeta^{k+1}), \zeta^{k+1} - \zeta \rangle + \langle g(\zeta^{k+\frac{1}{2}}), \zeta^{k+\frac{1}{2}} - \zeta^{k+1} \rangle \\ &= \frac{1}{\alpha} B_\zeta(\zeta, \zeta^k) - \frac{1}{\alpha} B_\zeta(\zeta, \zeta^{k+1}) - \frac{1}{\alpha} B_\zeta(\zeta^{k+1}, \zeta^k) + \langle g(\zeta^{k+\frac{1}{2}}), \zeta^{k+\frac{1}{2}} - \zeta^{k+1} \rangle, \end{aligned}$$

where the last equality uses the definition of Bregman divergence $B_\zeta(\zeta, \check{\zeta}) = d_\zeta(\zeta) -$

$(d_\zeta(\check{\zeta}) + \langle \nabla d_\zeta(\check{\zeta}), \zeta - \check{\zeta} \rangle)$. Further, for all $\zeta \in Q$,

$$\begin{aligned}
& \langle g(\zeta^{k+\frac{1}{2}}), \zeta^{k+\frac{1}{2}} - \zeta^{k+1} \rangle - \frac{1}{\alpha} B_\zeta(\zeta^{k+1}, \zeta^k) \\
&= \langle g(\zeta^{k+\frac{1}{2}}) - g(\zeta^k), \zeta^{k+\frac{1}{2}} - \zeta^{k+1} \rangle - \frac{1}{\alpha} B_\zeta(\zeta^{k+1}, \zeta^k) + \langle g(\zeta^k), \zeta^{k+\frac{1}{2}} - \zeta^{k+1} \rangle \\
&\stackrel{(23) \text{ with } \zeta=\zeta^{k+1}}{\leq} \langle g(\zeta^{k+\frac{1}{2}}) - g(\zeta^k), \zeta^{k+\frac{1}{2}} - \zeta^{k+1} \rangle \\
&\quad + \frac{1}{\alpha} \langle \nabla d_\zeta(\zeta^k) - \nabla d_\zeta(\zeta^{k+\frac{1}{2}}), \zeta^{k+\frac{1}{2}} - \zeta^{k+1} \rangle - \frac{1}{\alpha} B_\zeta(\zeta^{k+1}, \zeta^k) \\
&= \langle g(\zeta^{k+\frac{1}{2}}) - g(\zeta^k), \zeta^{k+\frac{1}{2}} - \zeta^{k+1} \rangle - \frac{1}{\alpha} B(\zeta^{k+\frac{1}{2}}, \zeta^k) - \frac{1}{\alpha} B(\zeta^{k+1}, \zeta^{k+\frac{1}{2}}) \\
&\stackrel{\textcircled{1}}{\leq} \|g(\zeta^{k+\frac{1}{2}}) - g(\zeta^k)\|_{\zeta, *} \|\zeta^{k+\frac{1}{2}} - \zeta^{k+1}\|_\zeta - \frac{1}{2\alpha} \left(\|\zeta^{k+\frac{1}{2}} - \zeta^k\|_\zeta^2 + \|\zeta^{k+\frac{1}{2}} - \zeta^{k+1}\|_\zeta^2 \right) \\
&\stackrel{\textcircled{2}}{\leq} L_\zeta \|\zeta^{k+\frac{1}{2}} - \zeta^k\|_\zeta \|\zeta^{k+\frac{1}{2}} - \zeta^{k+1}\|_\zeta - \frac{L_\zeta}{2} \left(\|\zeta^{k+\frac{1}{2}} - \zeta^k\|_\zeta^2 + \|\zeta^{k+\frac{1}{2}} - \zeta^{k+1}\|_\zeta^2 \right) \leq 0,
\end{aligned}$$

where in $\textcircled{1}$ we used that $B_\zeta(\zeta, \check{\zeta}) \geq \frac{1}{2} \|\zeta - \check{\zeta}\|_\zeta^2$ and in $\textcircled{2}$ we substituted $\alpha = 1/L_\zeta$ and used that $g(\zeta)$ is L_ζ -Lipschitz-continuous.

Combining the above two inequalities and the choice $\alpha = 1/L_\zeta$, we obtain, for all $\zeta \in Q$ and $i \geq 0$,

$$\langle g(\zeta^{i+\frac{1}{2}}), \zeta^{i+\frac{1}{2}} - \zeta \rangle \leq L_\zeta B_\zeta(\zeta, \zeta^i) - L_\zeta B_\zeta(\zeta, \zeta^{i+1}).$$

Summing up these inequalities for i from 0 to $k-1$, we have:

$$\sum_{i=0}^{k-1} \langle g(\zeta^{i+\frac{1}{2}}), \zeta^{i+\frac{1}{2}} - \zeta \rangle \leq L_\zeta (B_\zeta(\zeta, \zeta^0) - B_\zeta(\zeta, \zeta^k)) \leq L_\zeta B_\zeta(\zeta, \zeta^0).$$

Now we use the connection between $S(\mathbf{u}, \mathbf{v})$ and the operator $g(\zeta)$. By convexity of $S(\mathbf{u}, \mathbf{v})$ in \mathbf{u} and concavity of $S(\mathbf{u}, \mathbf{v})$ w.r.t \mathbf{v} , we have, for all $\mathbf{u} \in \mathcal{A}_U$,

$$\frac{1}{k} \sum_{i=0}^{k-1} \langle \nabla_{\mathbf{u}} S(\mathbf{u}^i, \mathbf{v}^i), \mathbf{u}^i - \mathbf{u} \rangle \geq \frac{1}{k} \sum_{i=0}^{k-1} (S(\mathbf{u}^i, \mathbf{v}^i) - S(\mathbf{u}, \mathbf{v}^i)) \geq \frac{1}{k} \sum_{i=0}^{k-1} S(\mathbf{u}^i, \mathbf{v}^i) - S(\mathbf{u}, \widehat{\mathbf{v}}^k).$$

In the same way, we obtain, for all $\mathbf{v} \in \mathcal{A}_V$,

$$\frac{1}{k} \sum_{i=0}^{k-1} \langle -\nabla_{\mathbf{v}} S(\mathbf{u}^i, \mathbf{v}^i), \mathbf{v}^i - \mathbf{v} \rangle \geq -\frac{1}{k} \sum_{i=0}^{k-1} S(\mathbf{u}^i, \mathbf{v}^i) + S(\widehat{\mathbf{u}}^k, \mathbf{v}).$$

Summing these inequalities, by the definition of $g(\zeta)$ we obtain that, for all $\mathbf{u} \in \mathcal{A}_U$, $\mathbf{v} \in \mathcal{A}_V$,

$$S(\widehat{\mathbf{u}}^k, \mathbf{v}) - S(\mathbf{u}, \widehat{\mathbf{v}}^k) \leq \frac{1}{k} \sum_{i=0}^{k-1} \langle g(\zeta^{i+\frac{1}{2}}), \zeta^{i+\frac{1}{2}} - \mathbf{u} \rangle \leq \frac{L_\zeta}{k} B_\zeta(\zeta, \zeta^0).$$

It remains to deduce the target accuracy in the value of function F and to

the solution accuracy of problem (14). Let $\hat{\mathbf{u}}^N = ((\hat{\mathbf{x}}^N)^\top, (\hat{\mathbf{p}}^N)^\top, (\hat{\mathbf{s}}^N)^\top)^\top$, $\hat{\mathbf{v}}^N = ((\hat{\mathbf{y}}^N)^\top, (\hat{\mathbf{q}}^N)^\top, (\hat{\mathbf{z}}^N)^\top)^\top$. For all $\mathbf{x} \in \mathcal{X}$, $\mathbf{p} \in \mathcal{P}$, $\mathbf{y} \in \mathcal{Y}$, $\mathbf{q} \in \mathcal{Q}$, $\mathbf{s} \in B_{R_S}(0)$, $\mathbf{z} \in B_{R_Z}(0)$, we have

$$\begin{aligned} & F(\hat{\mathbf{x}}^N, \hat{\mathbf{p}}^N, \mathbf{y}, \mathbf{q}) + \langle \mathbf{z}, \mathbf{W}_x \hat{\mathbf{x}}^N \rangle + \langle \hat{\mathbf{s}}^N, \mathbf{W}_y \mathbf{y} \rangle - F(\mathbf{x}, \mathbf{p}, \hat{\mathbf{y}}^N, \hat{\mathbf{q}}^N) - \langle \hat{\mathbf{z}}^N, \mathbf{W}_x \mathbf{x} \rangle - \langle \mathbf{s}, \mathbf{W}_y \hat{\mathbf{y}}^N \rangle \\ & \leq \frac{L_\zeta B_\zeta(\zeta, \zeta^0)}{N}. \end{aligned}$$

To get an upper bound on consensus residual, we substitute $\mathbf{x} = \hat{\mathbf{x}}^N$, $\mathbf{y} = \hat{\mathbf{y}}^N$, $\mathbf{p} = \hat{\mathbf{p}}^N$, $\mathbf{q} = \hat{\mathbf{q}}^N$, $\mathbf{z} = \hat{\mathbf{z}}^N + \frac{\mathbf{W}_x \hat{\mathbf{x}}^N R_Z}{\|\mathbf{W}_x \hat{\mathbf{x}}^N\|_2}$, $\mathbf{s} = 0$ and get

$$\begin{aligned} & R_Z \left\langle \frac{\mathbf{W}_x \hat{\mathbf{x}}^N}{\|\mathbf{W}_x \hat{\mathbf{x}}^N\|}, \mathbf{W}_x \hat{\mathbf{x}}^N \right\rangle \\ & \leq \frac{L_\zeta}{N} \left(\frac{B_x(\mathbf{x}^N, \mathbf{x}^0)}{R_x^2} + \frac{B_y(\mathbf{y}^N, \mathbf{y}^0)}{R_y^2} + \frac{B_p(\mathbf{p}^N, \mathbf{p}^0)}{R_p^2} + \frac{B_q(\mathbf{q}^N, \mathbf{q}^0)}{R_q^2} \right) \\ & \quad + \frac{L_\zeta}{2NR_Z^2} \left\| \hat{\mathbf{z}}^N + \frac{\mathbf{W}_x \hat{\mathbf{x}}^N R_Z}{\|\mathbf{W}_x \hat{\mathbf{x}}^N\|} \right\|_2^2 \\ & \leq \frac{4L_\zeta}{N} + \frac{L_\zeta}{N} \left(\frac{\|\hat{\mathbf{z}}^N\|_2^2}{R_Z^2} + 1 \right) \stackrel{\textcircled{1}}{\leq} \frac{17L_\zeta}{N}. \end{aligned}$$

Here in $\textcircled{1}$ we bounded $\|\hat{\mathbf{z}}^N\|_2$ the following way:

$$\begin{aligned} \|\hat{\mathbf{z}}^N\|_2^2 & \leq 2R_Z^2 B_\zeta(\hat{\zeta}^N, \zeta^0) \leq 2R_Z^2 B_\zeta(\zeta^*, \zeta^0) \\ & \leq 2R_Z^2 \left(\frac{B_x(\mathbf{x}^*, \mathbf{x}^0)}{R_x^2} + \frac{B_y(\mathbf{y}^*, \mathbf{y}^0)}{R_y^2} + \frac{B_p(\mathbf{p}^*, \mathbf{p}^0)}{R_p^2} + \frac{B_q(\mathbf{q}^*, \mathbf{q}^0)}{R_q^2} + \frac{B_S(\mathbf{s}^*, \mathbf{s}^0)}{R_S^2} + \frac{B_Z(\mathbf{z}^*, \mathbf{z}^0)}{R_Z^2} \right) \\ & \leq 12R_Z^2. \end{aligned}$$

As a result, we get

$$\|\mathbf{W}_x \hat{\mathbf{x}}^N\|_2 \leq \frac{17L_\zeta}{NR_Z} = \frac{17L_\zeta}{N} \frac{\lambda_{\min}^+(\mathbf{W}_x)}{M_x \sqrt{2}} = \frac{17\sqrt{2}L_\zeta}{2NM_x} \lambda_{\min}^+(\mathbf{W}_x).$$

Analogously we have

$$\|\mathbf{W}_y \hat{\mathbf{y}}^N\|_2 \leq \frac{17L_\zeta}{NR_S} = \frac{17L_\zeta}{N} \frac{\lambda_{\min}^+(\mathbf{W}_y)}{M_y \sqrt{2}} = \frac{17\sqrt{2}L_\zeta}{2NM_y} \lambda_{\min}^+(\mathbf{W}_y).$$

Let us now estimate the duality gap in objective function. Recall that

$$\begin{aligned} & F(\hat{\mathbf{x}}^N, \hat{\mathbf{p}}^N, \mathbf{y}, \mathbf{q}) + \langle \mathbf{z}, \mathbf{W}_x \hat{\mathbf{x}}^N \rangle + \langle \hat{\mathbf{s}}^N, \mathbf{W}_y \mathbf{y} \rangle - F(\mathbf{x}, \mathbf{p}, \hat{\mathbf{y}}^N, \hat{\mathbf{q}}^N) - \langle \hat{\mathbf{z}}^N, \mathbf{W}_x \mathbf{x} \rangle - \langle \mathbf{s}, \mathbf{W}_y \hat{\mathbf{y}}^N \rangle \\ & \leq \frac{L_\zeta}{N} \left(\frac{B_x(\mathbf{x}, \mathbf{x}^0)}{R_x^2} + \frac{B_p(\mathbf{p}, \mathbf{p}^0)}{R_p^2} + \frac{B_y(\mathbf{y}, \mathbf{y}^0)}{R_y^2} + \frac{B_q(\mathbf{q}, \mathbf{q}^0)}{R_q^2} + \frac{B_Z(\mathbf{z}, \mathbf{z}^0)}{R_Z^2} + \frac{B_S(\mathbf{s}, \mathbf{s}^0)}{R_S^2} \right). \end{aligned}$$

Also recall that $B_{\mathcal{T}}(\mathbf{t}, \mathbf{t}^0) \leq R_{\mathcal{T}}^2$ for $\mathbf{t} \in \{\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}\}$ and $\mathcal{T} \in \{\mathcal{X}, \mathcal{P}, \mathcal{Y}, \mathcal{Q}\}$, respectively.

Setting $\mathbf{z} = \mathbf{z}^0 = 0$, $\mathbf{s} = \mathbf{s}^0 = 0$ we obtain

$$F(\widehat{\mathbf{x}}^N, \widehat{\mathbf{p}}^N, \mathbf{y}, \mathbf{q}) + \langle \widehat{\mathbf{s}}, \mathbf{W}_y \mathbf{y} \rangle - F(\mathbf{x}, \mathbf{p}, \widehat{\mathbf{y}}^N, \widehat{\mathbf{q}}^N) - \langle \widehat{\mathbf{z}}, \mathbf{W}_x \mathbf{x} \rangle \leq \frac{4L_\zeta}{N}.$$

Note that

$$\begin{aligned} \max_{\mathbf{W}_y \mathbf{y}=0} [F(\widehat{\mathbf{x}}^N, \widehat{\mathbf{p}}^N, \mathbf{y}, \mathbf{q}) + \langle \widehat{\mathbf{s}}^N, \mathbf{W}_y \mathbf{y} \rangle] &= \max_{\mathbf{W}_y \mathbf{y}=0} F(\widehat{\mathbf{x}}^N, \widehat{\mathbf{p}}^N, \mathbf{y}, \mathbf{q}), \\ \min_{\mathbf{W}_x \mathbf{x}=0} [F(\mathbf{x}, \mathbf{p}, \widehat{\mathbf{y}}^N, \widehat{\mathbf{q}}^N) + \langle \widehat{\mathbf{z}}^N, \mathbf{W}_x \mathbf{x} \rangle] &= \min_{\mathbf{W}_x \mathbf{x}=0} F(\mathbf{x}, \mathbf{p}, \widehat{\mathbf{y}}^N, \widehat{\mathbf{q}}^N). \end{aligned}$$

Then

$$\max_{\mathbf{W}_y \mathbf{y}=0} F(\widehat{\mathbf{x}}^N, \widehat{\mathbf{p}}^N, \mathbf{y}, \mathbf{q}) - \min_{\mathbf{W}_x \mathbf{x}=0} F(\mathbf{x}, \mathbf{p}, \widehat{\mathbf{y}}^N, \widehat{\mathbf{q}}^N) \leq \frac{4L_\zeta}{N}. \quad (25)$$

In the criterion above, the max and min operations are taken over consensus sets $\mathbf{W}_x \mathbf{x} = 0$ and $\mathbf{W}_y \mathbf{y} = 0$, but the variables $\widehat{\mathbf{x}}, \widehat{\mathbf{y}}$ may not satisfy consensus constraints. The criterion can be updated as follows.

Introduce $\widetilde{\mathbf{x}}^N = \frac{1}{m} \mathbf{1} \mathbf{1}^\top \widehat{\mathbf{x}}^N$, $\widetilde{\mathbf{y}}^N = \frac{1}{m} \mathbf{1} \mathbf{1}^\top \widehat{\mathbf{y}}^N$. Let us recall that gradients of F are uniformly bounded (see Assumption 3.1 of the paper). Using convexity of F w.r.t \mathbf{x} we obtain

$$\begin{aligned} F(\widetilde{\mathbf{x}}^N, \widehat{\mathbf{p}}^N, \mathbf{y}, \mathbf{q}) &\leq F(\widehat{\mathbf{x}}^N, \widehat{\mathbf{p}}^N, \mathbf{y}, \mathbf{q}) - \langle \nabla F_{\mathbf{x}}(\widetilde{\mathbf{x}}^N, \widehat{\mathbf{p}}^N, \mathbf{y}, \mathbf{q}), \widehat{\mathbf{x}}^N - \widetilde{\mathbf{x}}^N \rangle \\ &\leq F(\widehat{\mathbf{x}}^N, \widehat{\mathbf{p}}^N, \mathbf{y}, \mathbf{q}) + \|\nabla F_{\mathbf{x}}(\widetilde{\mathbf{x}}^N, \widehat{\mathbf{p}}^N, \mathbf{y}, \mathbf{q})\| \cdot \|\widehat{\mathbf{x}}^N - \widetilde{\mathbf{x}}^N\|_2 \\ &\leq F(\widehat{\mathbf{x}}^N, \widehat{\mathbf{p}}^N, \mathbf{y}, \mathbf{q}) + \sqrt{m} M_x \cdot \frac{\|\mathbf{W}_x(\widehat{\mathbf{x}}^N - \widetilde{\mathbf{x}}^N)\|_2}{\lambda_{\min}^+(\mathbf{W}_x)} \\ &\stackrel{\textcircled{1}}{\leq} F(\widehat{\mathbf{x}}^N, \widehat{\mathbf{p}}^N, \mathbf{y}, \mathbf{q}) + \frac{17L_\zeta \sqrt{m}}{N} \frac{M_x}{R_Z \lambda_{\min}^+(\mathbf{W}_x)} \\ &\stackrel{\textcircled{2}}{=} F(\widehat{\mathbf{x}}^N, \widehat{\mathbf{p}}^N, \mathbf{y}, \mathbf{q}) + \frac{17L_\zeta}{\sqrt{2}N}. \end{aligned}$$

Here in $\textcircled{1}$ we used the bound on consensus violation from Theorem 3.6 of the paper and in $\textcircled{2}$ we recalled the definition of R_Z from Lemma 3.2. Carrying out an analogical estimate of $F(\mathbf{x}, \mathbf{p}, \widetilde{\mathbf{y}}^N, \widehat{\mathbf{q}}^N)$, we obtain

$$\max_{\mathbf{W}_y \mathbf{y}=0} F(\widetilde{\mathbf{x}}^N, \widehat{\mathbf{p}}^N, \mathbf{y}, \mathbf{q}) - \min_{\mathbf{W}_x \mathbf{x}=0} F(\mathbf{x}, \mathbf{p}, \widetilde{\mathbf{y}}^N, \widehat{\mathbf{q}}^N) \leq \frac{(4 + 17\sqrt{2})L_\zeta}{N}.$$

3.4. Discussion for the Euclidean case

Theorem 3.5 ensures a $O(1/\varepsilon)$ convergence guarantee but still does not provide explicit dependencies on the network characteristics. We derive these dependencies in the simplified Euclidean case.

Corollary 3.7. *Let $L \triangleq \max\{L_{(\mathbf{x}, \mathbf{p})(\mathbf{x}, \mathbf{p})}, L_{(\mathbf{x}, \mathbf{p})(\mathbf{y}, \mathbf{q})}, L_{(\mathbf{y}, \mathbf{q})(\mathbf{x}, \mathbf{p})}, L_{(\mathbf{y}, \mathbf{q})(\mathbf{y}, \mathbf{q})}\}$, $M \triangleq \max\{M_x, M_y\}$ and $\chi = \max\{\chi(W_x), \chi(W_y)\}$. Let us also upper bound the sizes of the sets: for $\mathcal{T}_i \in \{\bar{\mathcal{X}}, \mathcal{P}_i, \bar{\mathcal{Y}}, \mathcal{Q}_i\}$, let $R_{\mathcal{T}_i}^2 \leq R^2$. For $\mathbf{t} \in \{\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}\}$ let $\mathbf{t} = (t_1^\top, \dots, t_m^\top)^\top$*

and introduce the average values $\widehat{t}_{av}^N = \frac{1}{m} \sum_{i=1}^m \widehat{t}_i^N$. Then after N iterations of Algorithm 1 it holds

$$\begin{aligned} \max_{\substack{y \in \mathcal{Y}, \\ \mathbf{q} \in \mathcal{Q}}} \frac{1}{m} \sum_{i=1}^m f_i(\widehat{x}_{av}^N, \widehat{p}_i^N, y, q_i) - \min_{\substack{x \in \mathcal{X}, \\ \mathbf{p} \in \mathcal{P}}} \frac{1}{m} \sum_{i=1}^m f_i(x, p_i, \widehat{y}_{av}^N, \widehat{q}_i^N) &= O\left(\max\left\{\frac{LR^2}{N}, \frac{MR\chi}{N}\right\}\right), \\ \left(\frac{1}{m} \sum_{i=1}^m \|\widehat{x}_i^N - \widehat{x}_{av}^N\|_2^2\right)^{1/2} &= O\left(\max\left\{\frac{LR^2}{MN}, \frac{R\chi}{N}\right\}\right), \\ \left(\frac{1}{m} \sum_{i=1}^m \|\widehat{y}_i^N - \widehat{y}_{av}^N\|_2^2\right)^{1/2} &= O\left(\max\left\{\frac{LR^2}{MN}, \frac{R\chi}{N}\right\}\right). \end{aligned}$$

Corollary 3.7 states the convergence rate in terms of number of iterations of Algorithm 1. Every iteration corresponds to one communication round and one oracle call per node, resulting in equivalent oracle and communication complexities.

Remark 1. Let assumptions of Corollary 3.7 be satisfied. Additionally, let the critical point of each $f_i(x_i, p_i, y_i, q_i)$ lie in the set $\mathcal{X} \times \mathcal{P}_i \times \mathcal{Y} \times \mathcal{Q}_i$ ($i = 1, \dots, m$). Then we have $M \leq LR$. The Chebyshev acceleration technique (see i.e. [37]) makes it possible to reduce χ to $O(1)$ at the cost of performing $\sqrt{\chi}$ communication rounds at each iteration. As a result, Algorithm 1 requires $O(LR^2/\varepsilon)$ oracle calls per node and $O(LR^2\sqrt{\chi}/\varepsilon)$ communication rounds yield a point $(\widehat{\mathbf{x}}, \widehat{\mathbf{p}}, \widehat{\mathbf{y}}, \widehat{\mathbf{q}})$ such that

$$\max_{\substack{y \in \mathcal{Y}, \\ \mathbf{q} \in \mathcal{Q}}} \frac{1}{m} \sum_{i=1}^m f_i(\widehat{x}_{av}^N, \widehat{p}_i^N, y, q_i) - \min_{\substack{x \in \mathcal{X}, \\ \mathbf{p} \in \mathcal{P}}} \frac{1}{m} \sum_{i=1}^m f_i(x, p_i, \widehat{y}_{av}^N, \widehat{q}_i^N) \leq \varepsilon.$$

As will be shown in Section 4, these oracle and communication complexities are optimal.

4. Lower bounds for distributed saddle point problems

In this section we present results on lower bounds. In obtaining them, we focus on the Euclidean case. Before proceeding directly to the lower bounds, we need to identify the class of algorithms for which they are valid. To describe this class of first-order methods, we use a similar definition of Black-Box procedure as in [51]. We assume that one local iteration costs t time units, and the communication round costs τ time units. Additionally, information can be transmitted only along the undirected edge of the network. Communications and local updates can take place in parallel and asynchronously. More formally, it can be described as follows.

Assumption 4.1. Each machine i can collect past values for x, p and y, q in internal memories $H_{i,T}^x \subseteq \mathcal{R}^{d_x}, H_{i,T}^p \subseteq \mathcal{R}^{d_{p_i}}, H_{i,T}^y \subseteq \mathcal{R}^{d_y}, H_{i,T}^q \subseteq \mathcal{R}^{d_{q_i}}$ at time $T \geq 0$.

Initialization. We assume that all nodes start with $x^0 = 0, p_i^0 = 0, y^0 = 0, q_i^0 = 0$, then $H_{i,0}^x = \{0\}, H_{i,0}^p = \{0\}, H_{i,0}^y = \{0\}, H_{i,0}^q = \{0\}$ for all i .

Connection. The internal memories are updated either by communications $\bar{H}_{i,T}^x, \bar{H}_{i,T}^y$ or local steps $\widehat{H}_{i,T}^x, \widehat{H}_{i,T}^p, \widehat{H}_{i,T}^y, \widehat{H}_{i,T}^q$:

$$H_{i,T}^x \subseteq \bar{H}_{i,T}^x \cup \widehat{H}_{i,T}^x, \quad H_{i,T}^p \subseteq \widehat{H}_{i,T}^p, \quad H_{m,T}^y \subseteq \bar{H}_{m,T}^y \cup \widehat{H}_{m,T}^y, \quad H_{i,T}^q \subseteq \widehat{H}_{i,T}^q.$$

Communication. Each machine i can start to communicate with neighbors at time $T - \tau$, then, after τ units of time, the update of memories using communications can be rewritten as follows:

$$\bar{H}_{i,T}^x \subseteq \text{span} \left\{ \bigcup_{(j,i) \in \mathcal{E}} H_{j,T-\tau}^x \right\}, \quad \bar{H}_{i,T}^y \subseteq \text{span} \left\{ \bigcup_{(j,i) \in \mathcal{E}} H_{j,T-\tau}^y \right\}.$$

Local computation. Locally, each device i can find $\nabla_x f_i(x, p, y, q)$, $\nabla_{p_i} f_i(x, p, y, q)$, $\nabla_y f_i(x, p, y, q)$, $\nabla_{q_i} f_i(x, p, y, q)$ for all $x \in H_{i,T-t}^x, p \in H_{i,T-t}^p, y \in H_{i,T-t}^y, q \in H_{i,T-t}^q$. Then, the local update is

$$\begin{aligned} \hat{H}_{i,T}^x &\subseteq \text{span} \{x, \nabla_x f_i(x', p', y', q'), \text{Mirr}(\gamma \nabla_x f_i(x', p', y', q'); x; \mathcal{X})\}, \\ \hat{H}_{i,T}^p &\subseteq \text{span} \{p, \nabla_{p_i} f_i(x', p', y', q'), \text{Mirr}(\gamma \nabla_{p_i} f_i(x', p', y', q'); p; \mathcal{P}_i)\}, \\ \hat{H}_{i,T}^y &\subseteq \text{span} \{y, \nabla_y f_i(x', p', y', q'), \text{Mirr}(\gamma \nabla_y f_i(x', p', y', q'); y; \mathcal{Y})\}, \\ \hat{H}_{i,T}^q &\subseteq \text{span} \{q, \nabla_{q_i} f_i(x', p', y', q'), \text{Mirr}(\gamma \nabla_{q_i} f_i(x', p', y', q'); q; \mathcal{Q}_i)\}, \end{aligned}$$

for all $x, x' \in H_{i,T-t}^x, p, p' \in H_{i,T-t}^p, y, y' \in H_{i,T-t}^y, q, q' \in H_{i,T-t}^q$ and all $\gamma \in \mathbb{R}$. Mirr is defined above in (15).

Output. When the algorithm ends (after T_0 units of time), we have m local outputs $x_i^f \in H_{i,T_0}^x, p_i^f \in H_{i,T_0}^p, y_i^f \in H_{i,T_0}^y, q_i^f \in H_{i,T_0}^q$. Suppose the final global output is calculated as follows:

$$x^f \in \text{span} \left\{ \bigcup_{i=1}^m H_{i,T_0}^x \right\}, \quad p_i^f \in H_{i,T_0}^p, \quad y^f \in \text{span} \left\{ \bigcup_{i=1}^m H_{i,T_0}^y \right\}, \quad q_i^f \in H_{i,T_0}^q.$$

Note that the above procedure do not communicate on variables p_i and q_i , because they are local and unique for each node, in contrast with variables x and y . It is easy to check that Algorithm 1 also satisfies the definition of the procedure above.

The idea of proving any lower bounds for optimization problems comes from the first results on lower bounds for convex minimization [46]. The essence is to give an example of a "bad" function. In the case of distributed problems, it is also necessary to "badly" divide this function between m devices [51]. We consider a special case of problem (1) (the variables \mathbf{p} and \mathbf{q} are dummies, and the sets \mathcal{P} and \mathcal{Q} are empty):

$$\min_{x \in \bar{\mathcal{X}}} \max_{y \in \bar{\mathcal{Y}}} f(x, y) = \frac{1}{m} \sum_{i=1}^m f_i(x, y), \quad (26)$$

where $\bar{\mathcal{X}}, \bar{\mathcal{Y}} \subset \mathbb{R}^d$ are balls with the centers at point 0 and some size R (see (16)). We will define the size R and n more precisely in the proof (see Appendix 9). Since we concentrate entirely on the Euclidean case, therefore the operator Mirr in the procedure definition above can be replaced by the Euclidean projection operator: $\text{Mirr}(g, x, X) = \text{proj}_X(x - g)$ with $\text{proj}_X(y) = \arg \min_{z \in X} \|z - y\|_2$.

The global function $f(x, y)$ in (26) is exactly an example of a bilinear function from the work on non-distributed lower bounds for the strongly convex-strongly concave case [60], but with strong convexity and strong concavity constants $\mu \sim \frac{\varepsilon}{R^2}$ (see (34) in Appendix 9). In fact, if we write the dual function for this $f(x, y)$ from (34), we have

exactly the problem from [46], but with a smoothness constant $\sim \frac{L^2}{\mu}$ (see Lemma 9.2 in Appendix 9). The problem (26) is decomposed into summands f_i in such a way that if we want to get closer to the solution, we need to communicate (see also [51]). This idea is explained in more detail in Appendix 9. Here we only give a simplified version of the main theorem.

Theorem 4.2. *For any given $L > 0$, and $\chi \geq 1$, there exist*

- *functions f_i defined on $\mathcal{X} \times \mathcal{P}_i \times \mathcal{Y} \times \mathcal{Q}_i$ ($i = 1, \dots, m$), such that they satisfy Assumption 2.1, the averaged function $f = \frac{1}{m} \sum_{i=1}^m f_i$ is L -smooth and all the sets $\mathcal{X} \times \mathcal{P}_i \times \mathcal{Y} \times \mathcal{Q}_i$ have size R ,*
- *a connected graph \mathcal{G} and a gossip matrix W associated with it. The condition number of the gossip matrix equals $\chi(W) = \chi$.*

Then for any decentralized algorithm satisfying Assumption 4.1 the minimal time T for yielding a ε -solution $(\hat{x}, \hat{\mathbf{p}}, \hat{y}, \hat{\mathbf{q}})$ (i.e. $\max_{y \in \mathcal{Y}, \mathbf{q} \in \mathcal{Q}} f(\hat{x}, \hat{\mathbf{p}}, y, \mathbf{q}) - \min_{x \in \mathcal{X}, \mathbf{p} \in \mathcal{P}} f(x, \mathbf{p}, \hat{y}, \hat{\mathbf{q}}) \leq \varepsilon$)

is given by

$$T = \Omega \left(\frac{LR^2}{\varepsilon} (t + \sqrt{\chi\tau}) \right).$$

Theorem 4.2 shows that the minimal number of oracle calls per node is $\Omega(LR^2/\varepsilon)$ and the minimal number of communication rounds is $\Omega(LR^2\sqrt{\chi(W)}/\varepsilon)$. In Remark 1 we showed that in the euclidean case Algorithm 1 with Chebyshev acceleration reaches these lower bounds when the critical point of each $f_i(x_i, p_i, y_i, q_i)$ lies in $\mathcal{X} \times \mathcal{P}_i \times \mathcal{Y} \times \mathcal{Q}_i$ ($i = 1, \dots, m$).

5. Application to the Wasserstein barycenters

Now we show the benefits of representing some convex problems as convex-concave problems on the example of the Wasserstein barycenter (WB) problem and solve it by the DMP algorithm. Similarly to Section (3), we consider a SPP in proximal setup and introduce Lagrangian multipliers for the common variables. However, in the Section 3 we obtained results in a general setup without additional knowledge about cost functions and sets. On the contrary, in this section we utilize the special structure of the WB problem and introduce slightly different norms. After that, we get a convergence guarantee by applying Theorem 3.5.

The (fixed-support) WB of probability measures y_1, y_2, \dots, y_m from the probability simplex Δ_n is a solution of the following finite-dimensional optimization problem

$$\min_{x \in \Delta_n} \frac{1}{m} \sum_{i=1}^m \mathcal{W}(x, y_i), \quad (27)$$

where $\mathcal{W}(x, y) = \min_{\pi \in U(x, y)} \langle C, \pi \rangle$ is optimal transport between two histograms $x, y \in \Delta_n$, $C \in \mathbb{R}_+^{n \times n}$ is a given ground cost matrix, and π is a transport plan which belongs to the transportation polytope $U = \{\pi \in \mathbb{R}_+^{n \times n}, \pi \mathbf{1} = x, \pi^\top \mathbf{1} = y\}$. As we consider a general cost matrix C , the optimal transport problem is more general than the problem defining a Wasserstein distance.

5.1. Saddle point problem formulation

Next we reformulate the WB problem (27) as a saddle point problem [17]. To do so, we introduce stacked column vector $b_i = [x_i^\top, y_i^\top]^\top$, vectorized cost matrix d of C , vectorized transport plan $p \in \Delta_{n^2}$ of π , the incidence matrix $A = \{0, 1\}^{2n \times n^2}$, vectors $q_i \in [-1, 1]^{2n}$ for all $i = 1, \dots, m$. Then the problem (27) can be equivalently rewritten as

$$\min_{\substack{x \in \Delta_n, \\ p_1, \dots, p_m \in \Delta_{n^2}}} \max_{q_1, \dots, q_m \in [-1, 1]^{2n}} \frac{1}{m} \sum_{i=1}^m f_i(x, p_i, q_i) \triangleq d^\top p + 2\|d\|_\infty \left(q_i^\top A p_i - b_i^\top q_i \right). \quad (28)$$

Clearly, this problem fits the template (1). Then we can rewrite it for stacked column vectors up to a multiplicative term $1/m$ to fit the template (14)

$$\min_{\mathbf{p} \in \mathcal{P}, \mathbf{x} \in \mathcal{X}} \max_{\mathbf{q} \in \mathcal{Q}} F(\mathbf{p}, \mathbf{x}, \mathbf{q}) \triangleq \mathbf{d}^\top \mathbf{p} + 2\|d\|_\infty \left(\mathbf{q}^\top \mathbf{A} \mathbf{p} - \mathbf{b}^\top \mathbf{q} \right) \text{ s.t.: } x_1 = x_2 = \dots = x_m. \quad (29)$$

where $\mathbf{x} = [x_1^\top \in \Delta_n, \dots, x_m^\top \in \Delta_n]^\top \in \mathcal{X} \triangleq \prod^m \Delta_n$ (here $\prod^m \Delta_n$ is the Cartesian product of m simplices), $\mathbf{p} = [p_1^\top \in \Delta_{n^2}, \dots, p_m^\top \in \Delta_{n^2}]^\top \in \mathcal{P} \triangleq \prod^m \Delta_{n^2}$, and $\mathbf{q} = [q_1^\top, \dots, q_m^\top]^\top \in \mathcal{Q} \triangleq [-1, 1]^{2nm}$, $\mathbf{b} = [x_1^\top, y_1^\top, \dots, x_m^\top, y_m^\top]^\top$, $\mathbf{d} = [d^\top, \dots, d^\top]^\top$ and $\mathbf{A} = \text{diag}\{A, \dots, A\} \in \{0, 1\}^{2mn \times mn^2}$ is the block-diagonal matrix.

Finally, to enable distributed computation of problem (29), we replace the constraint $x_1 = \dots = x_m$ by $\mathbf{W}\mathbf{x} = 0$ (matrix $\mathbf{W} \triangleq W \otimes I_n$, where W is defined in (2)). Then we introduce the Lagrangian dual variable $\mathbf{z} = [z_1^\top, \dots, z_m^\top]^\top \in \mathcal{Z} \triangleq \mathbb{R}^{nm}$ corresponding to the constraint $\mathbf{W}\mathbf{x} = 0$ and rewrite the problem (29) as follows

$$\min_{\mathbf{u} \in \mathcal{U}} \max_{\mathbf{v} \in \mathcal{A}_v} S(\mathbf{u}, \mathbf{v}) \triangleq \mathbf{d}^\top \mathbf{p} + 2\|d\|_\infty \left(\mathbf{q}^\top \mathbf{A} \mathbf{p} - \mathbf{b}^\top \mathbf{q} \right) + \langle \mathbf{z}, \mathbf{W}\mathbf{x} \rangle, \quad (30)$$

where $\mathbf{u} = (\mathbf{x}, \mathbf{p})$, $\mathbf{v} = (\mathbf{q}, \mathbf{z})$ and $\mathcal{U} \triangleq \mathcal{X} \times \mathcal{P}$, $\mathcal{A}_v \triangleq \mathcal{Q} \times \mathbb{R}^{nm}$. Problem (30) fits the form (17), hence it can be solved by the DMP.

5.2. Convergence guarantee

Before presenting the complexity bound for the DMP algorithm to solve the SPP (30), we firstly bound the norm of dual variable \mathbf{z} similarly to the general setup in Section 3.

Lemma 5.1. *Let $R_{\mathcal{Z}}^2 = 8mn \max_{i,j} C_{ij}^2 / (\lambda_{\min}^+(W))^2$. Then there exists a saddle point $(\mathbf{x}^*, \mathbf{p}^*, \mathbf{q}^*, \mathbf{z}^*)$ of problem (30) such that $\|\mathbf{z}^*\|_2 \leq R_{\mathcal{Z}}$.*

This lemma is essentially a particular case of Lemma 3.1 for the WB problem.

Lemma 5.2. *Objective $S(\mathbf{u}, \mathbf{v})$ in (30) is $(L_{\mathbf{u}\mathbf{u}}, L_{\mathbf{u}\mathbf{v}}, L_{\mathbf{v}\mathbf{u}}, L_{\mathbf{v}\mathbf{v}})$ -smooth with $L_{\mathbf{u}\mathbf{u}} = L_{\mathbf{v}\mathbf{v}} = 0$ and $L_{\mathbf{u}\mathbf{v}} = L_{\mathbf{v}\mathbf{u}} = 8m\sqrt{2n} \ln n \max_{i,j} C_{ij} \chi$ w.r.t. norms*

$$\|\mathbf{u}\|_{\mathcal{U}}^2 = \frac{1}{m \ln n} \sum_{i=1}^m \left(\|x_i\|_1^2 + \frac{1}{2} \|p_i\|_1^2 \right) \quad \text{and} \quad \|\mathbf{v}\|_{\mathcal{V}}^2 = \frac{1}{mn} \left(\|\mathbf{q}\|_2^2 + \frac{(\lambda_{\min}^+(W))^2 \|\mathbf{z}\|_2^2}{8 \max_{i,j} C_{ij}^2} \right).$$

Then similarly to (19) L_ζ can be defined as follows

$$L_\zeta = 2 \max\{L_{\mathbf{uu}}, L_{\mathbf{uv}}, L_{\mathbf{vu}}, L_{\mathbf{vv}}\} = 16m\sqrt{2n \ln n} \max_{i,j} C_{ij}\chi. \quad (31)$$

These two Lemmas proved in Appendix 10 together with (31) make it possible to obtain the convergence guarantee for the DMP algorithm (see Algorithm 3 in Appendix 11).

Theorem 5.3. *After $N = (4 + 17\sqrt{2})L_\zeta/(m\varepsilon)$ iterations, Algorithm 3 with stepsize $\alpha = 1/L_\zeta$, where L_ζ is defined in (31), yields output $\widehat{\mathbf{x}}^N = [(\widehat{x}_1^N)^\top, \dots, (\widehat{x}_m^N)^\top]^\top$ such that*

$$\frac{1}{m} \sum_{i=1}^m \mathcal{W}(\widehat{x}_{av}^N, y_i) - \min_{x \in \Delta_n} \frac{1}{m} \sum_{i=1}^m \mathcal{W}(x, y_i) \leq \varepsilon \quad \text{and} \quad \|\mathbf{W}\widehat{\mathbf{x}}^N\|_2 = O\left(\frac{\sqrt{m}\lambda_{\min}^+(W)\varepsilon}{\sqrt{n} \max_{i,j} C_{ij}}\right),$$

where $\widehat{x}_{av}^N = \frac{1}{m} \sum_{i=1}^m \widehat{x}_i^N$. The per node complexity of Algorithm 3 is

$$O\left(\frac{n^2}{\varepsilon} \sqrt{n \ln n} \max_{i,j} C_{ij}^2 \chi\right).$$

The proof follows from Theorem 3.5 and is given in the Appendix 10.

5.2.1. Discussion of the convergence results.

We comment on the complexity of the DMP algorithm compared to the existing state-of-the-art methods: the iterative Bregman projections (IBP) algorithm, its accelerated versions and primal dual algorithm (ADCWB), see Table 1. All of these methods use entropic regularization of Wasserstein metric with parameter γ which must be taken proportionally to accuracy ε .

Table 1.: Distributed algorithms for the WB problem

Algorithm	Per node complexity ⁽¹⁾
IBP [7, 33]	$n^2/(\gamma\varepsilon)$
Accelerated IBP [26]	$n^2\sqrt{n}/\sqrt{\gamma\varepsilon}$
FastIBP [38]	$n^2\sqrt[3]{n}/(\gamma\varepsilon)^{2/3}$
ADCWB [16, 18]	$n^2\sqrt{n}/\sqrt{\gamma\varepsilon}$
DMP (this work)	$n^2\sqrt{n}/\varepsilon$

⁽¹⁾ The bounds are obtained by using the Chebyshev acceleration.

5.3. Numerical experiments

In this section, we tested the DMP algorithm for computing WB (see Algorithm 3 in the Appendix 11) on Gaussians measures and the *notMNIST* dataset.

5.3.1. Evaluation on different network architectures.

We demonstrate the performance of the DMP algorithm on different network architectures with different conditional number χ : complete graph, star graph, cycle graph and the Erdős-Rényi random graphs with the probability of edge creation $p = 0.5$ and $p = 0.4$ under the random seed = 10. As the true barycenter of Gaussian measures can be calculated theoretically [14], we use them to study the convergence of the DMP to the non-optimality gap. We randomly generated 10 Gaussian measures with equally spaced support of 30 points in $[-10, -10]$, mean from $[-5, 5]$ and variance from $[0.8, 1.8]$. Figure 1 supports Theorem 5.3 and presents the convergence of the DMP to the function non-optimality gap and the distance to the consensus. The smaller the condition number, the faster the convergence.

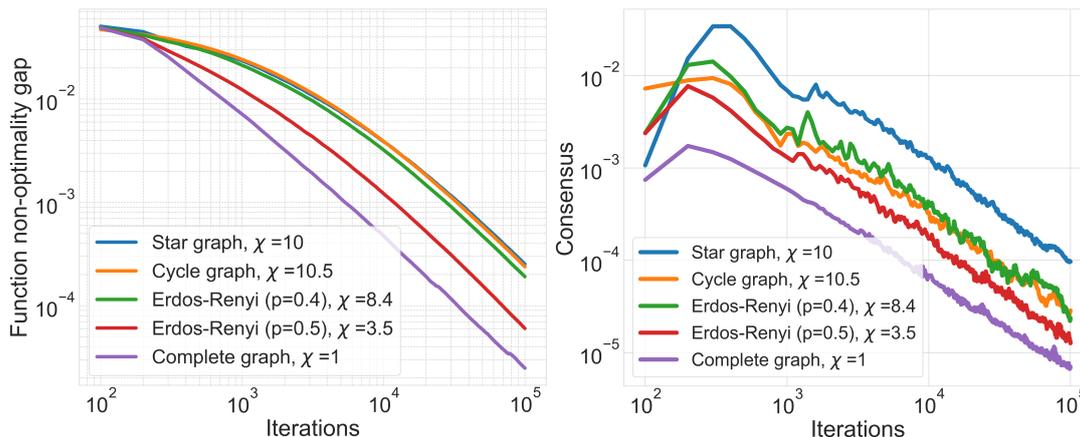


Figure 1.: Convergence of the DMP algorithm on different network architectures.

5.3.2. Comparison with existing methods.

Next we compare the DMP with the most popular modern algorithms: the IBP [7] and the ADCWB [18, Algorithm 4]. We illustrate a well-known issue: numerical instability of regularized algorithms with a small value of parameter γ to solve the WB problem. We ran the IBP and the ADCWB algorithms with different values of the regularization parameter γ starting from $\gamma = 0.1$ and gradually decreasing its value to $\gamma = 10^{-4}$. The number of iterations was taken proportionally to $1/\gamma$ in the IBP and proportionally to $1/\sqrt{\gamma}$ in the ADCWB according to the theoretical bounds. Figure 2 shows that for a certain value of γ (depending on the the experiment set and the number of method iterations) the regularized algorithms diverge. Our unregularized DMP algorithm is capable to achieve any accuracy, the more iteration the better accuracy. We ran it to achieve about 10^{-8} accuracy, probably the machine accuracy.

We also illustrate the non-stability of the IBP and the ADCWB algorithms run with $\gamma = 10^{-4}$ on the *notMNIST* dataset, in particular for the letter ‘B’ presented in various fonts. Figure 3 shows the best barycenters before the regularized algorithms will diverge.

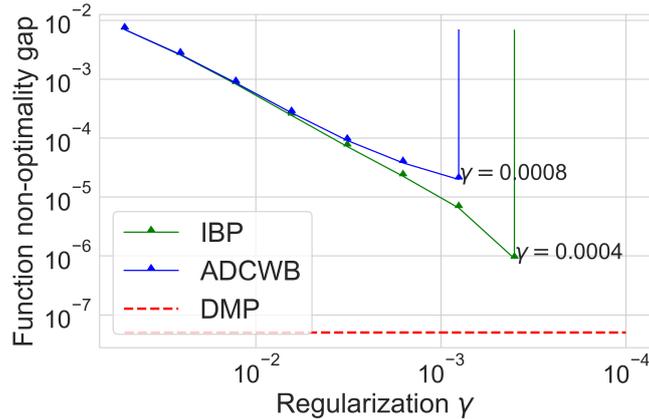


Figure 2.: Numerical instability of entropy-regularized based methods



Figure 3.: Barycenters of letter ‘B’ found by the DMP (left), the IBP (middle) and the ADCWB (right).

6. Conclusion

We proposed a decentralized method for saddle point problems based on non-Euclidean Mirror-Prox algorithm. Our reformulation is built upon moving the consensus constraints into the problem by adding Lagrangian multipliers. As a result, we get a common saddle point problem that includes both primal and dual variables. After that, we employ the Mirror-Prox algorithm and bound the norms of dual variables at solution to assist the theoretical analysis. Finally, we demonstrate the effectiveness of our approach on the problem of computing Wasserstein barycenters (both theoretically and numerically).

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7. Supplementary for Section 2

Define a sequence of Chebyshev polynomials as $T_0(\beta) = 1$, $T_1(\beta) = \beta$ and $T_{k+1}(\beta) = 2\beta T_k(\beta) - T_{k-1}(\beta)$ for $k \geq 1$. After that, let $c_1 = \frac{\sqrt{\chi}-1}{\sqrt{\chi}+1}$, $c_2 = \frac{\chi+1}{\chi-1}$, $c_3 = \frac{2}{\lambda_{\max}(\tilde{W}) + \lambda_{\min}^+(\tilde{W})}$ and introduce

$$P_K(\beta) = 1 - \frac{T_K(c_2(1-\beta))}{T_K(c_2)}.$$

Now let us consider a new communication matrix $P_K(\tilde{\mathbf{W}})$ with $K = \lfloor \sqrt{\chi} \rfloor$. As shown in [2], the spectrum of $P_K(\tilde{\mathbf{W}})$ lies in $\left[1 - \frac{2c_1^K}{1+c_1^{2K}}; 1 + \frac{2c_1^K}{1+c_1^{2K}}\right]$ and therefore $\lambda_{\max}(P_K(\tilde{\mathbf{W}})) = O(1)$, $\lambda_{\min}^+(P_K(\tilde{\mathbf{W}})) = O(1)$ and $\chi(P_K(\tilde{\mathbf{W}})) = O(1)$.

8. Smoothness constants for Mirror-Prox

8.1. Estimating Lipschitz constants for $S(\mathbf{u}, \mathbf{v})$

We start by deriving the Lipschitz constants $L_{\mathbf{u}\mathbf{u}}, L_{\mathbf{v}\mathbf{v}}, L_{\mathbf{u}\mathbf{v}}, L_{\mathbf{v}\mathbf{u}}$ of the function $S(\mathbf{u}, \mathbf{v})$ in (17). Recall that for an operator A which acts from a space V with a norm $\|v\|_v$ to a space W with a norm $\|w\|_w$, then these two norms naturally induce a norm of the operator A as $\|A\|_{v \rightarrow w} = \max_{v,w} \{\|Av\|_w : \|v\|_v \leq 1\}$, which gives an inequality $\|Av\|_w \leq \|A\|_{v \rightarrow w} \|v\|_v$. Recall also that (17) is a reformulation of (14) in a simpler form using the definitions $\mathbf{u} = (\mathbf{x}^\top, \mathbf{p}^\top, \mathbf{s}^\top)^\top$, $\mathbf{v} = (\mathbf{y}^\top, \mathbf{q}^\top, \mathbf{z}^\top)^\top$ and

$$S(\mathbf{u}, \mathbf{v}) = F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) + \langle \mathbf{s}, \mathbf{W}_y \mathbf{y} \rangle + \langle \mathbf{z}, \mathbf{W}_x \mathbf{x} \rangle.$$

Then for the corresponding partial derivatives we have

$$\nabla_{\mathbf{u}} S(\mathbf{u}, \mathbf{v}) = \begin{bmatrix} \nabla_{\mathbf{x}} F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) + \mathbf{W}_x \mathbf{z} \\ \nabla_{\mathbf{p}} F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) \\ \mathbf{W}_y \mathbf{y} \end{bmatrix}, \quad \nabla_{\mathbf{v}} S(\mathbf{u}, \mathbf{v}) = \begin{bmatrix} \nabla_{\mathbf{y}} F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) + \mathbf{W}_y \mathbf{s} \\ \nabla_{\mathbf{q}} F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) \\ \mathbf{W}_x \mathbf{x} \end{bmatrix}.$$

Let $\mathcal{A}_{\mathbf{u}} = \mathcal{X} \times \mathcal{P} \times \mathbb{R}^{nd_q}$, $\mathcal{A}_{\mathbf{v}} = \mathcal{Y} \times \mathcal{Q} \times \mathbb{R}^{nd_p}$ and $\mathcal{U} = \mathcal{X} \times \mathcal{P} \times \{\mathbf{s} : \|\mathbf{s}\|_2 \leq R_{\mathcal{S}}\}$, $\mathcal{V} = \mathcal{Y} \times \mathcal{Q} \times \{\mathbf{z} : \|\mathbf{z}\|_2 \leq R_{\mathcal{Z}}\}$. Recall that we introduced norms $\|\mathbf{u}\|_{\mathcal{U}}^2 = \|\mathbf{x}\|_{\mathcal{X}}^2 / R_{\mathcal{X}}^2 + \|\mathbf{p}\|_{\mathcal{P}}^2 / R_{\mathcal{P}}^2 + \|\mathbf{s}\|_2^2 / R_{\mathcal{S}}^2$, $\|\mathbf{v}\|_{\mathcal{V}}^2 = \|\mathbf{y}\|_{\mathcal{Y}}^2 / R_{\mathcal{Y}}^2 + \|\mathbf{q}\|_{\mathcal{Q}}^2 / R_{\mathcal{Q}}^2 + \|\mathbf{z}\|_2^2 / R_{\mathcal{Z}}^2$, which induce the dual norms $\|\mathbf{u}\|_{\mathcal{U},*}^2 = R_{\mathcal{X}}^2 \|\mathbf{x}\|_{\mathcal{X},*}^2 + R_{\mathcal{P}}^2 \|\mathbf{p}\|_{\mathcal{P},*}^2 + R_{\mathcal{S}}^2 \|\mathbf{s}\|_2^2$, $\|\mathbf{v}\|_{\mathcal{V},*}^2 = R_{\mathcal{Y}}^2 \|\mathbf{y}\|_{\mathcal{Y},*}^2 + R_{\mathcal{Q}}^2 \|\mathbf{q}\|_{\mathcal{Q},*}^2 + R_{\mathcal{Z}}^2 \|\mathbf{z}\|_2^2$. The constant $L_{\mathbf{u}\mathbf{u}}$ has to satisfy $\|\nabla_{\mathbf{u}} S(\mathbf{u}, \mathbf{v}) - \nabla_{\mathbf{u}} S(\mathbf{u}', \mathbf{v})\|_{\mathcal{U},*} \leq L_{\mathbf{u}\mathbf{u}} \|\mathbf{u} - \mathbf{u}'\|_{\mathcal{U}}$ for all $\mathbf{u}, \mathbf{u}' \in \mathcal{A}_{\mathbf{u}}$. We have

$$\begin{aligned} \|\nabla_{\mathbf{u}} S(\mathbf{u}, \mathbf{v}) - \nabla_{\mathbf{u}} S(\mathbf{u}', \mathbf{v})\|_{\mathcal{U},*}^2 &= \left\| \begin{bmatrix} \nabla_{\mathbf{x}} F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) - \nabla_{\mathbf{x}} F(\mathbf{x}', \mathbf{p}', \mathbf{y}, \mathbf{q}) \\ \nabla_{\mathbf{p}} F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) - \nabla_{\mathbf{p}} F(\mathbf{x}', \mathbf{p}', \mathbf{y}, \mathbf{q}) \\ 0 \end{bmatrix} \right\|_{\mathcal{U},*}^2 \\ &= R_{\mathcal{X}}^2 \|\nabla_{\mathbf{x}} F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) - \nabla_{\mathbf{x}} F(\mathbf{x}', \mathbf{p}', \mathbf{y}, \mathbf{q})\|_{\mathcal{X},*}^2 \\ &\quad + R_{\mathcal{P}}^2 \|\nabla_{\mathbf{p}} F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) - \nabla_{\mathbf{p}} F(\mathbf{x}', \mathbf{p}', \mathbf{y}, \mathbf{q})\|_{\mathcal{P},*}^2 \\ &\leq (R_{\mathcal{X}}^2 + R_{\mathcal{P}}^2) \left\| \begin{bmatrix} \nabla_{\mathbf{x}} F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) - \nabla_{\mathbf{x}} F(\mathbf{x}', \mathbf{p}', \mathbf{y}, \mathbf{q}) \\ \nabla_{\mathbf{p}} F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) - \nabla_{\mathbf{p}} F(\mathbf{x}', \mathbf{p}', \mathbf{y}, \mathbf{q}) \end{bmatrix} \right\|_{(\mathcal{X}, \mathcal{P}),*}^2 \\ &\leq L_{(\mathbf{x}, \mathbf{p})(\mathbf{x}, \mathbf{p})}^2 (R_{\mathcal{X}}^2 + R_{\mathcal{P}}^2) \left\| \begin{bmatrix} \mathbf{x} - \mathbf{x}' \\ \mathbf{p} - \mathbf{p}' \end{bmatrix} \right\|_{(\mathcal{X}, \mathcal{P})}^2 \\ &= L_{(\mathbf{x}, \mathbf{p})(\mathbf{x}, \mathbf{p})}^2 (R_{\mathcal{X}}^2 + R_{\mathcal{P}}^2) \left(R_{\mathcal{X}}^2 \frac{\|\mathbf{x} - \mathbf{x}'\|_{\mathcal{X}}^2}{R_{\mathcal{X}}^2} + R_{\mathcal{P}}^2 \frac{\|\mathbf{p} - \mathbf{p}'\|_{\mathcal{P}}^2}{R_{\mathcal{P}}^2} \right) \\ &\leq L_{(\mathbf{x}, \mathbf{p})(\mathbf{x}, \mathbf{p})}^2 (R_{\mathcal{X}}^2 + R_{\mathcal{P}}^2)^2 \|\mathbf{u} - \mathbf{u}'\|_{\mathcal{U}}^2, \end{aligned}$$

where we used that $\|(\mathbf{x}, \mathbf{p})\|_{(\mathcal{X}, \mathcal{P}),*}^2 = \|\mathbf{x}\|_{\mathcal{X},*}^2 + \|\mathbf{p}\|_{\mathcal{P},*}^2$ since $\|(\mathbf{x}, \mathbf{p})\|_{(\mathcal{X}, \mathcal{P})}^2 = \|\mathbf{x}\|_{\mathcal{X}}^2 + \|\mathbf{p}\|_{\mathcal{P}}^2$, Assumption 3.3. Thus, $L_{\mathbf{u}\mathbf{u}} = L_{(\mathbf{x}, \mathbf{p})(\mathbf{x}, \mathbf{p})} (R_{\mathcal{X}}^2 + R_{\mathcal{P}}^2)$. The equality $L_{\mathbf{v}\mathbf{v}} = L_{(\mathbf{y}, \mathbf{q})(\mathbf{y}, \mathbf{q})} (R_{\mathcal{Y}}^2 + R_{\mathcal{Q}}^2)$ is proved in the same way.

Let us estimate $L_{\mathbf{u}\mathbf{v}}$, which has to satisfy $\|\nabla_{\mathbf{u}}S(\mathbf{u}, \mathbf{v}) - \nabla_{\mathbf{u}}S(\mathbf{u}, \mathbf{v}')\|_{\mathcal{U},*} \leq L_{\mathbf{u}\mathbf{v}}\|\mathbf{v} - \mathbf{v}'\|_{\mathcal{V}}$ for all $\mathbf{v}, \mathbf{v}' \in \mathcal{A}_{\mathcal{V}}$. We have

$$\begin{aligned}
& \|\nabla_{\mathbf{u}}S(\mathbf{u}, \mathbf{v}) - \nabla_{\mathbf{u}}S(\mathbf{u}, \mathbf{v}')\|_{\mathcal{U},*}^2 \\
&= \left\| \begin{bmatrix} \nabla_{\mathbf{x}}F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) + \mathbf{W}_{\mathbf{x}}\mathbf{z} - \nabla_{\mathbf{x}}F(\mathbf{x}, \mathbf{p}, \mathbf{y}', \mathbf{q}') - \mathbf{W}_{\mathbf{x}}\mathbf{z}' \\ \nabla_{\mathbf{p}}F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) - \nabla_{\mathbf{p}}F(\mathbf{x}, \mathbf{p}, \mathbf{y}', \mathbf{q}') \\ \mathbf{W}_{\mathbf{y}}\mathbf{y} - \mathbf{W}_{\mathbf{y}}\mathbf{y}' \end{bmatrix} \right\|_{\mathcal{U},*}^2 \\
&\stackrel{\textcircled{1}}{=} R_{\mathcal{X}}^2 \|\nabla_{\mathbf{x}}F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) - \nabla_{\mathbf{x}}F(\mathbf{x}, \mathbf{p}, \mathbf{y}', \mathbf{q}') + \mathbf{W}_{\mathbf{x}}(\mathbf{z} - \mathbf{z}')\|_{\mathcal{X},*}^2 \\
&\quad + R_{\mathcal{P}}^2 \|\nabla_{\mathbf{p}}F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) - \nabla_{\mathbf{p}}F(\mathbf{x}, \mathbf{p}, \mathbf{y}', \mathbf{q}')\|_{\mathcal{P},*}^2 + R_{\mathcal{S}}^2 \|\mathbf{W}_{\mathbf{y}}(\mathbf{y} - \mathbf{y}')\|_2^2 \\
&\stackrel{\textcircled{2}}{\leq} 2R_{\mathcal{X}}^2 \|\nabla_{\mathbf{x}}F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) - \nabla_{\mathbf{x}}F(\mathbf{x}, \mathbf{p}, \mathbf{y}', \mathbf{q}')\|_{\mathcal{X},*}^2 + 2R_{\mathcal{X}}^2 \|\mathbf{W}_{\mathbf{x}}(\mathbf{z} - \mathbf{z}')\|_{\mathcal{X},*}^2 \\
&\quad + R_{\mathcal{P}}^2 \|\nabla_{\mathbf{p}}F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) - \nabla_{\mathbf{p}}F(\mathbf{x}, \mathbf{p}, \mathbf{y}', \mathbf{q}')\|_{\mathcal{P},*}^2 + R_{\mathcal{S}}^2 \|\mathbf{W}_{\mathbf{y}}(\mathbf{y} - \mathbf{y}')\|_2^2 \\
&\stackrel{\textcircled{3}}{\leq} 2(R_{\mathcal{X}}^2 + R_{\mathcal{P}}^2) \left\| \begin{bmatrix} \nabla_{\mathbf{x}}F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) - \nabla_{\mathbf{x}}F(\mathbf{x}, \mathbf{p}, \mathbf{y}', \mathbf{q}') \\ \nabla_{\mathbf{p}}F(\mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}) - \nabla_{\mathbf{p}}F(\mathbf{x}, \mathbf{p}, \mathbf{y}', \mathbf{q}') \end{bmatrix} \right\|_{(\mathcal{X}, \mathcal{P}),*}^2 \\
&\quad + 2R_{\mathcal{X}}^2 \|\mathbf{W}_{\mathbf{x}}\|_{2 \rightarrow (\mathbf{x},*)}^2 \|\mathbf{z} - \mathbf{z}'\|_2^2 + R_{\mathcal{S}}^2 \|\mathbf{W}_{\mathbf{y}}\|_{\mathbf{y} \rightarrow 2}^2 \|\mathbf{y} - \mathbf{y}'\|_{\mathcal{Y}}^2 \\
&\stackrel{\textcircled{4}}{\leq} 2L_{(\mathbf{x}, \mathbf{p})(\mathbf{y}, \mathbf{q})}^2 (R_{\mathcal{X}}^2 + R_{\mathcal{P}}^2) \left\| \begin{bmatrix} \mathbf{y} - \mathbf{y}' \\ \mathbf{q} - \mathbf{q}' \end{bmatrix} \right\|_{(\mathcal{Y}, \mathcal{Q})}^2 + 2R_{\mathcal{X}}^2 \|\mathbf{W}_{\mathbf{x}}\|_{2 \rightarrow (\mathbf{x},*)}^2 \|\mathbf{z} - \mathbf{z}'\|_2^2 \\
&\quad + R_{\mathcal{S}}^2 \|\mathbf{W}_{\mathbf{y}}\|_{\mathbf{y} \rightarrow 2}^2 \|\mathbf{y} - \mathbf{y}'\|_{\mathcal{Y}}^2 \\
&\leq \left(2L_{(\mathbf{x}, \mathbf{p})(\mathbf{y}, \mathbf{q})}^2 (R_{\mathcal{X}}^2 + R_{\mathcal{P}}^2) (R_{\mathcal{Y}}^2 + R_{\mathcal{Q}}^2) + 2R_{\mathcal{X}}^2 R_{\mathcal{Z}}^2 \|\mathbf{W}_{\mathbf{x}}\|_{2 \rightarrow (\mathbf{x},*)}^2 \right. \\
&\quad \left. + 2R_{\mathcal{Y}}^2 R_{\mathcal{S}}^2 \|\mathbf{W}_{\mathbf{y}}\|_{\mathbf{y} \rightarrow 2}^2 \right) \|\mathbf{v} - \mathbf{v}'\|_{\mathcal{V}}^2,
\end{aligned}$$

where we used in $\textcircled{1}$ the definition of $\|\mathbf{u}\|_{\mathcal{U},*}^2$; in $\textcircled{2}$ the inequality $(a + b)^2 \leq 2a^2 + 2b^2$; in $\textcircled{3}$ that $\|(\mathbf{x}, \mathbf{p})\|_{(\mathbf{x}, \mathbf{p}),*}^2 = \|\mathbf{x}\|_{\mathcal{X},*}^2 + \|\mathbf{p}\|_{\mathcal{P},*}^2$ since $\|(\mathbf{x}, \mathbf{p})\|_{(\mathcal{X}, \mathcal{P}),*}^2 = \|\mathbf{x}\|_{\mathcal{X}}^2 + \|\mathbf{p}\|_{\mathcal{P}}^2$ and the definition of the operator norm; in $\textcircled{4}$ Assumption 3.3; and, finally, in $\textcircled{5}$ the definition of $\|\mathbf{v}\|_{\mathcal{V}}^2$. Taking the square root of the derived inequality, we obtain

$$L_{\mathbf{u}\mathbf{v}} \leq \sqrt{2} \max \left\{ L_{(\mathbf{x}, \mathbf{p})(\mathbf{y}, \mathbf{q})} \sqrt{(R_{\mathcal{X}}^2 + R_{\mathcal{P}}^2)(R_{\mathcal{Y}}^2 + R_{\mathcal{Q}}^2)}, \right. \\
\left. R_{\mathcal{X}} R_{\mathcal{Z}} \|\mathbf{W}_{\mathbf{x}}\|_{2 \rightarrow (\mathbf{x},*)}, R_{\mathcal{Y}} R_{\mathcal{S}} \|\mathbf{W}_{\mathbf{y}}\|_{\mathbf{y} \rightarrow 2} \right\}.$$

The bound for $L_{\mathbf{v}\mathbf{u}}$ is derived in the same way:

$$L_{\mathbf{v}\mathbf{u}} \leq \sqrt{2} \max \left\{ L_{(\mathbf{y}, \mathbf{q})(\mathbf{x}, \mathbf{p})} \sqrt{(R_{\mathcal{X}}^2 + R_{\mathcal{P}}^2)(R_{\mathcal{Y}}^2 + R_{\mathcal{Q}}^2)}, \right. \\
\left. R_{\mathcal{X}} R_{\mathcal{Z}} \|\mathbf{W}_{\mathbf{x}}\|_{\mathbf{x} \rightarrow 2}, R_{\mathcal{Y}} R_{\mathcal{S}} \|\mathbf{W}_{\mathbf{y}}\|_{2 \rightarrow (\mathbf{y},*)} \right\}.$$

9. Proof of lower bounds from Theorem 4.2

Let $B \subseteq \mathcal{V}$ be a subset of nodes of \mathcal{G} . For $\rho > 0$ we define $B_{\rho} = \{v \in \mathcal{V} : \rho(B, v) \geq \rho\}$, where $\rho(B, v)$ is a distance between set B and node v . Then, we construct the following

B_ρ only the first coordinates of x and y can be non-zero, the rest coordinates are strictly zero. For the rest of the nodes, all coordinates remains strictly zero. Without communications, the situation does not change. Therefore, we need to make at least ρ communications in order to have non-zero first coordinates in some node from B (transfer of information from B_ρ to B). Using (33), by local iterations (at least 1) at the node of the set B , one can achieve the first and second non-zero coordinates. Next the process continues with respect of (33).

Hence, to get at least one node $i \in \mathcal{V}$ with the internal memory $H_i^x, H_i^y \subseteq \text{span}\{e_1, \dots, e_{k_0}\}$, we need a minimum of $k_0 + 1$ local steps (at least 2 steps in the beginning, when we start from 0, and then at least 1 local step in other cases – see previous paragraph), as well as $(k_0 - 1)\rho$ communication rounds. In other words,

$$\max_i \left[\max\{k \in \mathcal{N} : \exists x \in H_{i,T}^x, \exists y \in H_{i,T}^y \text{ s.t. } x_k \neq 0, y_k \neq 0\} \right] \leq \left\lfloor \frac{T - 2t}{t + \rho\tau} \right\rfloor + 2.$$

According to Assumption 4.1, we have that the final global output is the union of all the outputs. Whence the statement of Lemma holds. \square

The previous lemma gives us an understanding of how quickly we can approximate the solution. In particular, in coordinates that can be non-zero we are able to have a value that absolutely coincides with the solution, but in zero coordinates this is impossible. It remains to understand what the solution even looks like. Considering the global objective function, we have:

$$\begin{aligned} f(x, y) &= \frac{1}{M} \sum_{m=1}^M f_m(x, y) \\ &= \frac{1}{M} (|B_\rho| \cdot f_1(x, y) + |B| \cdot f_2(x, y) + (M - |B_\rho| - |B|) \cdot f_3(x, y)) \\ &= \frac{L}{2} x^T A y + \frac{16\varepsilon}{R^2} \|x\|_2^2 - \frac{16\varepsilon}{R^2} \|y\|_2^2 - \frac{L^2 R^2}{\varepsilon} e_1^T y, \quad \text{with } A = \frac{1}{2}(A_1 + A_2). \end{aligned} \quad (34)$$

With $\|A\| \leq 2$, it is easy to check that f_i is convex-concave and f is also $\max\{L, \frac{32\varepsilon}{R^2}\}$ -smooth.

Lemma 9.2 (Lemma 3.3 from [60]). *Let $\alpha = \frac{32\varepsilon^2}{L^2 R^4}$ and $q = \frac{1}{2} \left(2 + \alpha - \sqrt{\alpha^2 + 4\alpha} \right) \in (0; 1)$ – the smallest root of $q^2 - (2 + \alpha)q + 1 = 0$, and let introduce the approximation of the solution \bar{y}^* :*

$$\bar{y}_i^* = \frac{q^i}{1 - q}.$$

Then error between the approximation and the real solution of (34) can be bounded:

$$\|\bar{y}^* - y^*\|_2 \leq \frac{q^{d+1}}{\alpha(1 - q)}.$$

Proof. Let us write the dual function for (34):

$$g(y) = -\frac{1}{2} y^T \left(\frac{L^2 R^2}{\varepsilon} A^T A + \frac{32\varepsilon}{R^2} I \right) y + \frac{L^2 R^2}{\varepsilon} e_1^T y,$$

after T units of time:

$$\|x_T - x^*\|_2^2 + \|y_T - y^*\|_2^2 \geq q^{2\left(\frac{T}{t+\rho\tau}+2\right)} \frac{\|y_0 - y^*\|_2^2}{16}.$$

Proof. By Lemma 9.2 with $q < 1$ and $k \leq \frac{d}{2}$, we have

$$\begin{aligned} \|y_T - \bar{y}^*\|_2^2 &\geq \sqrt{\sum_{j=k+1}^d (\bar{y}_j^*)^2} = \frac{q^k}{1-q} \sqrt{q^2 + q^4 + \dots + q^{2(d-k)}} \\ &\geq \frac{q^k}{\sqrt{2}(1-q)} \sqrt{q^2 + q^4 + \dots + q^{2d}} = \frac{q^k}{\sqrt{2}} \|y^*\|_2^2 = \frac{q^k}{\sqrt{2}} \|y_0 - \bar{y}^*\|_2^2. \end{aligned}$$

For $d \geq 2 \log_q \left(\frac{\alpha}{4\sqrt{2}} \right)$ we can guarantee that $\bar{y}^* \approx y^*$ and also (for more details see [60])

$$\begin{aligned} \|x_T - x^*\|_2^2 + \|y_T - y^*\|_2^2 &\geq \|y_T - y^*\|_2^2 \geq \frac{q^{2k}}{16} \|y_0 - y^*\|_2^2 = q^{2\left(\lfloor \frac{T-2t}{t+\rho\tau} \rfloor + 2\right)} \frac{\|y_0 - y^*\|_2^2}{16} \\ &\geq q^{2\left(\frac{T}{t+\rho\tau}+2\right)} \frac{\|y_0 - y^*\|_2^2}{16}. \end{aligned}$$

□

In the conditions of Lemma 9.3 there is a choice of d . With given y^* we can also determine the size of the ball R . More precisely we want $y^* \in \bar{\mathcal{Y}}$, then we can choose $R = \frac{1}{\sqrt{2}} \|y^* - y^0\| = \frac{1}{\sqrt{2}} \|y^*\|$ (see (16)). Now we are ready to combine the facts obtained above and prove Theorem 4.2.

Theorem 9.4 (Theorem 4.2). *Let $L > 0$, $\chi \geq 1$, $\varepsilon > 0$. Additionally, we assume that $L \geq \frac{32\varepsilon}{R^2}$. There exists a distributed saddle point problem of m (defined in the proof) functions with decentralized architecture and a gossip matrix W , for which the following statements are valid:*

- each function $f_i : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ is convex-concave,
- $f = \frac{1}{m} \sum_{i=1}^m f_i : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ is L -smooth and has $y^* \neq 0$,
- $d \geq \max \left\{ 2 \log_q \left(\frac{\alpha}{4\sqrt{2}} \right), 2k \right\}$, where $k = \left\lfloor \frac{T-2t}{t+\tau} \right\rfloor + 2$, $\alpha = \frac{32\varepsilon^2}{L^2 R^4}$ and $q = \frac{1}{2} \left(2 + \alpha - \sqrt{\alpha^2 + 4\alpha} \right) \in (0; 1)$,
- $\bar{\mathcal{X}} \times \bar{\mathcal{Y}}$ is bounded and has a size $R = \frac{1}{\sqrt{2}} \|y^*\|$,
- the gossip matrix W have $\chi(W) = \chi$.

Then for any procedure, which satisfies Assumption 4.1, one can bounded the time to achieve a ε -solution (i.e. $\max_{y \in \bar{\mathcal{Y}}} f(x_T, y) - \min_{x \in \bar{\mathcal{X}}} f(x, y_T) \leq \varepsilon$) in the final global output:

$$T = \Omega \left(\frac{LR^2}{\varepsilon} (t + \sqrt{\chi\tau}) \right).$$

Proof. We start the proof from Lemma 9.3 and get:

$$\|x_T - x^*\|^2 + \|y_T - y^*\|^2 \geq q^{2\left(\frac{T}{t+\rho\tau}+2\right)} \frac{\|y_0 - y^*\|^2}{16}.$$

Using $\frac{32\varepsilon}{R^2}$ - strong convexity – strong concavity of (34), we can obtain

$$\begin{aligned} \max_{y \in \mathcal{Y}} f(x^T, y) - \min_{x \in \mathcal{X}} f(x, y^T) &\geq f(x_T, y^*) - f(x^*, y^*) + f(x^*, y^*) - f(x^*, y_T) \\ &\geq \frac{16\varepsilon}{R^2} \|x_T - x^*\|^2 + \frac{16\varepsilon}{R^2} \|y_T - y^*\|^2, \end{aligned}$$

and

$$\max_{y \in \mathcal{Y}} f(x^T, y) - \min_{x \in \mathcal{X}} f(x, y^T) \geq q^{2\left(\frac{T}{t+\rho\tau}+2\right)} \cdot \frac{\varepsilon}{R^2} \|y_0 - y^*\|^2 \geq q^{2\left(\frac{T}{t+\rho\tau}+2\right)} \cdot 2\varepsilon.$$

If $\max_{y \in \mathcal{Y}} f(x^T, y) - \min_{x \in \mathcal{X}} f(x, y^T) \leq \varepsilon$, we get

$$\frac{2T}{t + \rho\tau} = \Omega(\log_{q^{-1}}(2)). \quad (35)$$

The next steps of the proof follows similar way with the proof of [51, Theorem 2]. Let $\chi_m = \frac{1+\cos\frac{\pi}{m}}{1-\cos\frac{\pi}{m}}$ be an increasing sequence of positive numbers (this sequence is also the condition numbers for the Laplacian of a linear graph of m vertexes). Since $\gamma_2 = 1$ and $\lim_m \chi_m = +\infty$, there exists $m \geq 2$ such that $\chi_m \leq \chi < \chi_{m+1}$.

• If $m \geq 3$, let us consider a linear graph with m vertexes v_1, \dots, v_m and with weighted edges $w_{1,2} = 1 - a$ and $w_{i,i+1} = 1$ for $i \geq 2$. If W_a is the Laplacian of this weighted graph, one can note that with $a = 0$, $\chi(W_a) = \gamma_m$, with $a = 1$, we have $\chi(W_a) = +\infty$. Hence, there exists $a \in [0; 1)$ such that $\chi(W_a) = \chi$. Then $\chi < \chi_{m+1} \leq \frac{(m+1)^2}{2}$. Finally, if $B = \{v_1\}$, $\bar{B} = \{v_m\}$, $\rho = m - 1$ and then, $\rho \geq \sqrt{2\chi} - 2$. Using (35), we have

$$T = \Omega\left(\left(t + \sqrt{\chi}\tau\right) \frac{1}{\ln q^{-1}}\right).$$

• If $M = 2$, we construct a fully connected network with 3 nodes and a weight $w_{1,3} = a \in [0; 1]$. Let W_a is the Laplacian. If $a = 0$, then the network is a linear graph and $\chi(W_a) = \chi_3 = 3$. Hence, there exists $a \in [0; 1)$ such that $\chi(W_a) = \chi$. We take $B = \{v_1\}$, $\bar{B} = \{v_3\}$ and get $\rho \geq 1 \geq \frac{\chi}{3}$. Finally, we have the same estimate as in the previous point

$$T = \Omega\left(\left(t + \sqrt{\chi}\tau\right) \frac{1}{\ln q^{-1}}\right).$$

Next, we work with

$$\begin{aligned}
\frac{1}{\ln(q^{-1})} &= \frac{1}{\ln(1 + (1 - q)/q)} \geq \frac{q}{1 - q} \\
&= \frac{1 + \frac{\alpha}{2} - \sqrt{\frac{\alpha^2}{4} + \alpha}}{\sqrt{\frac{\alpha^2}{4} + \alpha} - \frac{\alpha}{2}} = \frac{\sqrt{\frac{\alpha^2}{4} + \alpha} - \frac{\alpha}{2}}{\alpha} \\
&= \sqrt{\frac{1}{4} + \frac{1}{\alpha}} - \frac{1}{2} = \sqrt{\frac{1}{4} + \frac{1}{32} \left(\frac{LR^2}{\varepsilon}\right)^2} - \frac{1}{2}.
\end{aligned}$$

Hence, we get

$$T = \Omega\left(\frac{LR^2}{\varepsilon}(t + \sqrt{\chi\tau})\right).$$

□

10. Missing proofs from Section 5

Proof of Lemma (5.1). From Lemma 3.1 it holds

$$R_{\mathcal{Z}}^2 = \frac{2mM_x^2}{(\lambda_{\min}^+(W))^2} \quad (36)$$

where positive scalar M_x is such that: $\|\nabla_x f_i(x, p_i, q_i)\|_2 \leq M_x$ and $f_i(x, p_i, q_i)$ is defined in (28). Next we calculate its gradient

$$\nabla_x f_i(x, p_i, q_i) = -2\|d\|_{\infty}[q_i]_{1\dots n},$$

where $[q_i]_{1\dots n}$ is the first n component of vector $q_i \in [-1, 1]^{2n}$. As $q_i \in [-1, 1]^{2n}$ we have

$$\|\nabla f_i(x, p_i, q_i)\|_2 = 2\|d\|_{\infty}\|[q_i]_{1\dots n}\|_2 \leq 2\|d\|_{\infty}\sqrt{n} = M_x. \quad (37)$$

Thus, using this and the definition of d which provides $\|d\|_{\infty} = \max_{i,j} C_{ij}$ for (36) we get

$$R_{\mathcal{Z}}^2 = \frac{8mn \max_{i,j} C_{ij}^2}{(\lambda_{\min}^+(W))^2}.$$

□

Proof of Lemma (5.2). As $S(\mathbf{u}, \mathbf{v})$ from (17) is bilinear, $L_{\mathbf{uu}} = L_{\mathbf{vv}} = 0$. Then from Lemma 3.4 it follows that $S(\mathbf{u}, \mathbf{v})$ is $(0, L_{\mathbf{uv}}, L_{\mathbf{vu}}, 0)$ -smooth, where

$$\begin{aligned}
L_{\mathbf{uv}} &= \sqrt{2} \left(L_{(\mathbf{x}, \mathbf{p})\mathbf{q}} \sqrt{(R_{\mathcal{X}}^2 + R_{\mathcal{P}}^2)R_{\mathcal{Q}}^2} + R_{\mathcal{X}}R_{\mathcal{Z}} \|\mathbf{W}\|_{2 \rightarrow (\mathbf{x}, *)} \right), \\
L_{\mathbf{vu}} &= \sqrt{2} \left(L_{\mathbf{q}(\mathbf{x}, \mathbf{p})} \sqrt{(R_{\mathcal{X}}^2 + R_{\mathcal{P}}^2)R_{\mathcal{Q}}^2} + R_{\mathcal{X}}R_{\mathcal{Z}} \|\mathbf{W}\|_{\mathbf{x} \rightarrow 2} \right).
\end{aligned} \quad (38)$$

As $x_i \in \Delta_n$ and $p_i \in \Delta_{n^2}$, we use the simplex setup: ℓ_1 -norm and the following prox-functions on Δ_n and Δ_{n^2}

$$d_{\Delta_n}(x_i) = \langle x_i, \log x_i \rangle \quad \text{and} \quad d_{\Delta_{n^2}}(p_i) = \langle p_i, \log p_i \rangle. \quad (39)$$

For $\mathcal{X} \triangleq \prod^m \Delta_n$ and $\mathcal{P} \triangleq \prod^m \Delta_{n^2}$, we choose the following ℓ_1/ℓ_2 -norms

$$\|\mathbf{x}\|_{\mathcal{X}}^2 = \sum_{i=1}^m \|x_i\|_1^2 \quad \text{and} \quad \|\mathbf{p}\|_{\mathcal{P}}^2 = \sum_{i=1}^m \|p_i\|_1^2. \quad (40)$$

Then from (16) we have

$$\begin{aligned} R_{\Delta_n}^2 &= \max_{x_i \in \Delta_n} d_{\Delta_n}(x_i) - \min_{x_i \in \Delta_n} d_{\Delta_n}(x_i) = \ln n, \\ R_{\Delta_{n^2}}^2 &= \max_{p_i \in \Delta_{n^2}} d_{\Delta_{n^2}}(p_i) - \min_{p_i \in \Delta_{n^2}} d_{\Delta_{n^2}}(p_i) = 2 \ln n. \end{aligned}$$

Then from this and definitions of $\mathcal{X} \triangleq \prod^m \Delta_n$ and $\mathcal{P} \triangleq \prod^m \Delta_{n^2}$, it follows

$$R_{\mathcal{X}}^2 = \sum_{i=1}^m R_{\Delta_n}^2 = m \ln n, \quad \text{and} \quad R_{\mathcal{P}}^2 = \sum_{i=1}^m R_{\Delta_{n^2}}^2 = 2m \ln n. \quad (41)$$

Using this and (39), we get

$$d_{\mathbf{u}}(\mathbf{u}) = \sum_{i=1}^m \left(\frac{d_{\Delta_n}(x_i)}{R_{\mathcal{X}}^2} + \frac{d_{\Delta_{n^2}}(p_i)}{R_{\mathcal{P}}^2} \right) = \frac{1}{m \ln n} \sum_{i=1}^m \left(\langle x_i, \ln x_i \rangle + \frac{1}{2} \langle p_i, \ln p_i \rangle \right) \quad (42)$$

From (18) and (40), it holds

$$\|\mathbf{u}\|_{\mathcal{U}}^2 = \frac{\|\mathbf{x}\|_{\mathcal{X}}^2}{R_{\mathcal{X}}^2} + \frac{\|\mathbf{p}\|_{\mathcal{P}}^2}{R_{\mathcal{P}}^2} = \frac{1}{m \ln n} \sum_{i=1}^m \left(\|x_i\|_1^2 + \frac{1}{2} \|p_i\|_1^2 \right)$$

Now as $\mathcal{A}_{\mathcal{Y}}$ is unbounded, we define $\mathcal{V} = \mathcal{Q} \times \{\mathbf{z} : \|\mathbf{z}\|_2 \leq R_{\mathcal{Z}}\}$. On \mathcal{V} we define the Euclidean prox-setup. Let $\mathcal{Q}_i \triangleq [-1, 1]^{2n}$. Then for $q_i \in \mathcal{Q}_i$ we define

$$d_{\mathcal{Q}_i}(q_i) = \frac{1}{2} \|q_i\|_2^2. \quad (43)$$

Then

$$R_{\mathcal{Q}}^2 = \sum_{i=1}^m R_{\mathcal{Q}_i}^2 = \frac{1}{2} \sum_{i=1}^m \max_{q_i \in \mathcal{Q}_i} \|q_i\|_2 - \min_{q_i \in \mathcal{Q}_i} \|q_i\|_2 = mn \quad (44)$$

Then using this, (43) and Lemma (5.1), we have

$$d_{\mathbf{v}}(\mathbf{v}) = \sum_{i=1}^m \frac{d_{\mathcal{Q}_i}(q_i)}{R_{\mathcal{Q}}^2} + \frac{d_{\mathcal{Z}_i}(z_i)}{R_{\mathcal{Z}}^2} = \frac{1}{2mn} \left(\|\mathbf{q}\|_2^2 + \frac{(\lambda_{\min}^+(W))^2 \|\mathbf{z}\|_2^2}{8 \max_{i,j} C_{ij}^2} \right). \quad (45)$$

From (18) we have

$$\|\mathbf{v}\|_{\mathcal{V}}^2 = \frac{\|\mathbf{q}\|_{\mathcal{Q}}^2}{R_{\mathcal{Q}}^2} + \frac{\|\mathbf{z}\|_2^2}{R_{\mathcal{Z}}^2} = \frac{1}{2mn} \left(\|\mathbf{q}\|_2^2 + \frac{(\lambda_{\min}^+(W))^2 \|\mathbf{z}\|_2^2}{8 \max_{i,j} C_{ij}^2} \right).$$

Thus, we use (41), (44) and rewrite Eq. (38)

$$\begin{aligned} L_{\mathbf{u}\mathbf{v}} &= m\sqrt{n \ln n} \left(\sqrt{6}L_{(\mathbf{x},\mathbf{p})\mathbf{q}} + \frac{4\|d\|_{\infty}}{\lambda_{\min}(W)} \|\mathbf{W}\|_{2 \rightarrow (\mathbf{x},*)} \right), \\ L_{\mathbf{v}\mathbf{u}} &= m\sqrt{n \ln n} \left(\sqrt{6}L_{\mathbf{q}(\mathbf{x},\mathbf{p})} + \frac{4\|d\|_{\infty}}{\lambda_{\min}(W)} \|\mathbf{W}\|_{\mathbf{x} \rightarrow 2} \right). \end{aligned} \quad (46)$$

Now we consider

$$\|\mathbf{W}\|_{\mathbf{x} \rightarrow 2}^2 = \max_{\|\mathbf{x}\|_{\mathcal{X}}^2 \leq 1} \|\mathbf{W}\mathbf{x}\|_2^2 = \max_{\sum_{i=1}^m \|x_i\|_1^2 \leq 1} \|\mathbf{W}\mathbf{x}\|_2^2. \quad (47)$$

The set $\sum_{i=1}^m \|x_i\|_1^2 \leq 1$ is contained in the set $\sum_{j=1}^n \sum_{i=1}^m [x_i]_j^2 \leq 1$ as cross-product terms of $\|x_i\|_1^2$ are non-negative. Thus, we can change the constraint in the minimum in Eq. (47) as follows

$$\begin{aligned} \max_{\sum_{i=1}^m \|x_i\|_1^2 \leq 1} \|\mathbf{W}\mathbf{x}\|_2^2 &\leq \max_{\sum_{j=1}^n \sum_{i=1}^m [x_i]_j^2 \leq 1} \|\mathbf{W}\mathbf{x}\|_2^2 = \max_{\|\mathbf{x}\|_2^2 \leq 1} \|\mathbf{W}\mathbf{x}\|_2^2 \\ &= \max_{\|\mathbf{x}\|_2^2 \leq 1} \|\mathbf{W}\mathbf{x}\|_2^2 \triangleq \lambda_{\max}^2(\mathbf{W}) = \lambda_{\max}^2(W). \end{aligned} \quad (48)$$

The last inequality holds due to $\mathbf{W} \triangleq W \otimes I_n$ and the properties of the Kronecker product for eigenvalues. Thus, $\|\mathbf{W}\|_{\mathbf{x} \rightarrow 2} = \lambda_{\max}^2(W)$.

Next, we consider

$$\begin{aligned} \|\mathbf{W}\|_{\mathbf{x} \rightarrow 2}^2 &= \max_{\|\mathbf{x}\|_2^2 \leq 1} \|\mathbf{W}\mathbf{x}\|_{\mathcal{X},*}^2 = \max_{\|\mathbf{x}\|_2^2 \leq 1} \sum_{i=1}^m \|Wx\|_{\infty}^2 \leq \max_{\|\mathbf{x}\|_2^2 \leq 1} \sum_{i=1}^m \|Wx\|_2^2 \\ &= \max_{\|\mathbf{x}\|_2^2 \leq 1} \|\mathbf{W}\mathbf{x}\|_2^2 = \lambda_{\max}^2(W), \end{aligned} \quad (49)$$

where the last equality holds due to (48). Next, we estimate $L_{(\mathbf{x},\mathbf{p})\mathbf{q}}$ and $L_{\mathbf{q}(\mathbf{x},\mathbf{p})}$ by the definition

$$\left\| \begin{bmatrix} \nabla_{\mathbf{x}} F(\mathbf{x}, \mathbf{p}, \mathbf{q}) - \nabla_{\mathbf{x}} F(\mathbf{x}, \mathbf{p}, \mathbf{q}') \\ \nabla_{\mathbf{p}} F(\mathbf{x}, \mathbf{p}, \mathbf{q}) - \nabla_{\mathbf{p}} F(\mathbf{x}, \mathbf{p}, \mathbf{q}') \end{bmatrix} \right\|_{(\mathcal{X},\mathcal{P}),*} \leq L_{(\mathbf{x},\mathbf{p})\mathbf{q}} \|\mathbf{q} - \mathbf{q}'\|_2 \quad (50)$$

From the definition of dual norm and $\|(\mathbf{x}, \mathbf{p})\|_{(\mathcal{X},\mathcal{P})}^2 = \|\mathbf{x}\|_{\mathcal{X}}^2 + \|\mathbf{p}\|_{\mathcal{P}}^2$, it follows

$$\begin{aligned} &\left\| \begin{bmatrix} \nabla_{\mathbf{x}} F(\mathbf{x}, \mathbf{p}, \mathbf{q}) - \nabla_{\mathbf{x}} F(\mathbf{x}, \mathbf{p}, \mathbf{q}') \\ \nabla_{\mathbf{p}} F(\mathbf{x}, \mathbf{p}, \mathbf{q}) - \nabla_{\mathbf{p}} F(\mathbf{x}, \mathbf{p}, \mathbf{q}') \end{bmatrix} \right\|_{(\mathcal{X},\mathcal{P}),*} \\ &= \max_{\|\mathbf{x}\|_{\mathcal{X}}^2 + \|\mathbf{p}\|_{\mathcal{P}}^2 \leq 1} \left\langle \begin{bmatrix} \mathbf{x} \\ \mathbf{p} \end{bmatrix}, \begin{bmatrix} \nabla_{\mathbf{x}} F(\mathbf{x}, \mathbf{p}, \mathbf{q}) - \nabla_{\mathbf{x}} F(\mathbf{x}, \mathbf{p}, \mathbf{q}') \\ \nabla_{\mathbf{p}} F(\mathbf{x}, \mathbf{p}, \mathbf{q}) - \nabla_{\mathbf{p}} F(\mathbf{x}, \mathbf{p}, \mathbf{q}') \end{bmatrix} \right\rangle. \end{aligned}$$

From this and (50) we get

$$\max_{\|\mathbf{x}\|_1^2 + \|\mathbf{1}\|_p^2 \leq 1} \left\langle \begin{bmatrix} \mathbf{x} \\ \mathbf{p} \end{bmatrix}, \begin{bmatrix} \nabla_{\mathbf{x}} F(\mathbf{x}, \mathbf{p}, \mathbf{q}) - \nabla_{\mathbf{x}} F(\mathbf{x}, \mathbf{p}, \mathbf{q}') \\ \nabla_{\mathbf{p}} F(\mathbf{x}, \mathbf{p}, \mathbf{q}) - \nabla_{\mathbf{p}} F(\mathbf{x}, \mathbf{p}, \mathbf{q}') \end{bmatrix} \right\rangle \leq L_{(\mathbf{x}, \mathbf{p})\mathbf{q}} \|\mathbf{q} - \mathbf{q}'\|_2. \quad (51)$$

By the definition of $F(\mathbf{x}, \mathbf{p}, \mathbf{q})$ we have

$$\begin{bmatrix} \nabla_{\mathbf{x}} F(\mathbf{x}, \mathbf{p}, \mathbf{q}) \\ \nabla_{\mathbf{p}} F(\mathbf{x}, \mathbf{p}, \mathbf{q}) \\ -\nabla_{\mathbf{q}} F(\mathbf{x}, \mathbf{p}, \mathbf{q}) \end{bmatrix} = \begin{bmatrix} -2\|d\|_\infty \{[q_i]_{1\dots n}\}_{i=1}^m \\ \mathbf{d} + 2\|d\|_\infty \mathbf{A}^\top \mathbf{q} \\ -2\|d\|_\infty (\mathbf{A}\mathbf{p} - \mathbf{b}) \end{bmatrix}$$

Here $\{[q_i]_{1\dots n}\}_{i=1}^m$ is a short form of vector $([q_1]_{1\dots n}, [q_2]_{1\dots n}, \dots, [q_m]_{1\dots n})$.

From this it follows that $\begin{bmatrix} \nabla_{\mathbf{x}} F(\mathbf{x}, \mathbf{p}, \mathbf{q}) \\ \nabla_{\mathbf{p}} F(\mathbf{x}, \mathbf{p}, \mathbf{q}) \end{bmatrix}$ is linear function in $\mathbf{q} - \mathbf{q}'$, then (51) can be rewritten as

$$L_{(\mathbf{x}, \mathbf{p})\mathbf{q}} = \max_{\|\mathbf{q} - \mathbf{q}'\|_2 \leq 1} \max_{\|\mathbf{x}\|_x^2 + \|\mathbf{p}\|_p^2 \leq 1} \left\langle \begin{bmatrix} \mathbf{x} \\ \mathbf{p} \end{bmatrix}, \begin{bmatrix} \nabla_{\mathbf{x}} F(\mathbf{x}, \mathbf{p}, \mathbf{q}) - \nabla_{\mathbf{x}} F(\mathbf{x}, \mathbf{p}, \mathbf{q}') \\ \nabla_{\mathbf{p}} F(\mathbf{x}, \mathbf{p}, \mathbf{q}) - \nabla_{\mathbf{p}} F(\mathbf{x}, \mathbf{p}, \mathbf{q}') \end{bmatrix} \right\rangle. \quad (52)$$

Then

$$\begin{aligned} \begin{bmatrix} \nabla_{\mathbf{x}} F(\mathbf{x}, \mathbf{p}, \mathbf{q}) - \nabla_{\mathbf{x}} F(\mathbf{x}, \mathbf{p}, \mathbf{q}') \\ \nabla_{\mathbf{p}} F(\mathbf{x}, \mathbf{p}, \mathbf{q}) - \nabla_{\mathbf{p}} F(\mathbf{x}, \mathbf{p}, \mathbf{q}') \end{bmatrix} &= \begin{bmatrix} -2\|d\|_\infty \{[q_i - q'_i]_{1\dots n}\}_{i=1}^m \\ 2\|d\|_\infty \mathbf{A}^\top (\mathbf{q} - \mathbf{q}') \end{bmatrix} \\ &= 2\|d\|_\infty \begin{pmatrix} -\mathcal{E} \\ \mathbf{A}^\top \end{pmatrix} (\mathbf{q} - \mathbf{q}'), \end{aligned}$$

where $\mathcal{E} \in \{1, 0\}^{mn \times 2mn}$ is a block-diagonal matrix

$$\mathcal{E} = \begin{bmatrix} [I_n \ 0_{n \times n}] & \cdots & 0_{2n \times n} \\ \vdots & \ddots & \vdots \\ 0_{2n \times n} & \cdots & [I_n \ 0_{n \times n}] \end{bmatrix}.$$

Thus, we use this for (52) and get

$$L_{(\mathbf{x}, \mathbf{p})\mathbf{q}} = \max_{\|\mathbf{x}\|_x^2 + \|\mathbf{p}\|_p^2 \leq 1} \max_{\|\mathbf{x}\|_1^2 + \|\mathbf{1}\|_p^2 \leq 1} \left\langle \begin{bmatrix} \mathbf{x} \\ \mathbf{p} \end{bmatrix}, 2\|d\|_\infty \begin{bmatrix} -\mathcal{E} \\ \mathbf{A}^\top \end{bmatrix} [\mathbf{q} - \mathbf{q}'] \right\rangle. \quad (53)$$

By the same arguments we can get the same expression for $L_{\mathbf{q}(\mathbf{x}, \mathbf{p})}$ up to rearrangement of maximums. Next, we use the fact that the ℓ_2 -norm is the conjugate norm for the ℓ_2 -norm. From this and (52) it follows

$$\begin{aligned} L_{(\mathbf{x}, \mathbf{p})\mathbf{q}} &= \max_{\|\mathbf{x}\|_x^2 + \|\mathbf{p}\|_p^2 \leq 1} 2\|d\|_\infty \left\| \begin{bmatrix} -\mathcal{E}^\top & \mathbf{A} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{p} \end{bmatrix} \right\|_2 \\ &= \max_{\|\mathbf{x}\|_x^2 + \|\mathbf{p}\|_p^2 \leq 1} 2\|d\|_\infty \left\| -\mathcal{E}^\top \mathbf{x} + \mathbf{A}\mathbf{p} \right\|_2 \end{aligned} \quad (54)$$

Then

$$\| -\mathcal{E}^\top \mathbf{x} + \mathbf{A}\mathbf{p} \|_2^2 = \sum_{i=1}^m \left\| -\begin{pmatrix} x_i \\ 0_n \end{pmatrix} + Ap_i \right\|_2^2 \leq \sum_{i=1}^m \|x_i\|_2^2 + \sum_{i=1}^m \|Ap_i\|_2^2. \quad (55)$$

The last bound holds due to $\langle Ap_i, (x_i^\top, 0_n^\top) \rangle \geq 0$ as the entries of $A, \mathbf{p}, \mathbf{x}$ are non-negative.

Next we use this for (54)

$$\begin{aligned} L_{(\mathbf{x}, \mathbf{p})\mathbf{q}}^2 &= 4\|d\|_\infty^2 \max_{\|\mathbf{x}\|_\chi^2 + \|\mathbf{p}\|_p^2 \leq 1} \|\mathbf{A}\mathbf{p} - \mathcal{E}\mathbf{x}\|_2^2 = \max_{\|\mathbf{x}\|_\chi^2 + \|\mathbf{p}\|_p^2 \leq 1} \|\mathbf{A}\mathbf{p} - \mathcal{E}\mathbf{x}\|_2^2 \\ &\stackrel{(55)}{\leq} 4\|d\|_\infty^2 \max_{\alpha \in \Delta_{2m}} \left(\sum_{i=1}^m \max_{\|p_i\|_1^2 \leq \alpha_i} \|Ap_i\|_2^2 + \sum_{i=1}^m \max_{\|x_i\|_1^2 \leq \alpha_{i+m}} \|x_i\|_2^2 \right) \\ &= 4\|d\|_\infty^2 \max_{\alpha \in \Delta_{2m}} \left(\sum_{i=1}^m \alpha_i \max_{\|p_i\|_1 \leq 1} \|Ap_i\|_2^2 + \sum_{i=1}^m \alpha_{i+m} \max_{\|x_i\|_1 \leq 1} \|x_i\|_2^2 \right). \end{aligned} \quad (56)$$

By the definition of incidence matrix A we get $Ap_i = (h_1^\top, h_2^\top)$, where h_1 and h_2 such that $\mathbf{1}^\top h_1 = \mathbf{1}^\top h_2 = \sum_{j=1}^{n^2} [p_i]_j = 1$ as $p_i \in \Delta_{n^2} \forall i = 1, \dots, m$. Thus,

$$\|Ap_i\|_2^2 = \|h_1\|_2^2 + \|h_2\|_2^2 \leq \|h_1\|_1^2 + \|h_2\|_1^2 = 2. \quad (57)$$

As $x_i \in \Delta_n, \forall i = 1, \dots, m$ we have

$$\max_{\|x_i\|_1 \leq 1} \|x_i\|_2^2 \leq \max_{\|x_i\|_1 \leq 1} \|x_i\|_1^2 = 1. \quad (58)$$

Using (57) and (58) in (56) we get

$$\begin{aligned} L_{(\mathbf{x}, \mathbf{p})\mathbf{q}}^2 &= 4\|d\|_\infty^2 \max_{\|\mathbf{p}\|_1^2 + \|\mathbf{x}\|_1^2 \leq 1} \|\mathbf{A}\mathbf{p} - \mathcal{E}\mathbf{x}\|_2^2 \\ &\leq 4\|d\|_\infty^2 \max_{\alpha \in \Delta_{2m}} m \ln n \left(2 \sum_{i=1}^m \alpha_i + \sum_{i=m+1}^{2m} \alpha_i \right) \leq \max_{\alpha \in \Delta_{2m}} 2 \sum_{i=1}^{2m} \alpha_i = 8\|d\|_\infty^2. \end{aligned} \quad (59)$$

Hence,

$$L_{(\mathbf{x}, \mathbf{p})\mathbf{q}} = L_{\mathbf{q}(\mathbf{x}, \mathbf{p})} = 2\sqrt{2}\|d\|_\infty. \quad (60)$$

Using this, (48) and (49) for (46) and we get

$$L_{\mathbf{u}\mathbf{v}} = L_{\mathbf{v}\mathbf{u}} = L_{\mathbf{v}\mathbf{u}} = 4m\sqrt{n \ln n} \|d\|_\infty (\sqrt{3} + \chi). \quad (61)$$

Then we use that for any $a, b, \sqrt{a^2 + b^2} \leq \sqrt{2} \max\{a, b\}$ and rewrite this as follows

$$L_{\mathbf{v}\mathbf{u}} = 4m\sqrt{n \ln n} \|d\|_\infty (\sqrt{3} + \chi) \leq 4m\sqrt{2n \ln n} \|d\|_\infty \max\{\sqrt{3}, \chi\} \leq 8m\sqrt{2n \ln n} \|d\|_\infty \chi.$$

The last holds due to $\chi \geq 1$ for any graph. Finally, we use $\|d\|_\infty = \max_{i,j} C_{ij}$ and finish the proof.

□

10.1. Proof of Theorem 5.3

Proof of Theorem 5.3. From the Theorem 3.5, it holds

$$\max_{\mathbf{q} \in \mathcal{Q}} \sum_{i=1}^m f_i(\widehat{x}_{av}^N, \widehat{p}_i^N, q_i) - \min_{\substack{x \in \mathcal{X}, \\ \mathbf{p} \in \mathcal{P}}} \sum_{i=1}^m f_i(x, p_i, \widehat{q}_i^N) \leq \frac{(4 + 17\sqrt{2})L_\zeta}{N} \quad (62)$$

where from (28)

$$f_i(x_i, p_i, q_i) \triangleq d^\top p_i + 2\|d\|_\infty \left(q_i^\top A p_i - b_i^\top q_i \right) \quad (63)$$

and

$$\|\mathbf{W}\widehat{\mathbf{x}}^N\|_2 \leq \frac{17L_\zeta}{NR_{\mathcal{Z}}}. \quad (64)$$

By the definition of optimal transport

$$\begin{aligned} \mathcal{W}(x_i, y_i) &= \min_{\pi \in U(x_i, y_i)} \langle C, \pi \rangle = \min_{A p_i = b_i, p_i \in \Delta_n} d^\top p_i = \min_{p_i \in \Delta_n} d^\top p_i + 2\|d\|_\infty \|A p_i - b_i\|_1, \\ &= \min_{p_i \in \Delta_n} \max_{q_i \in [-1, 1]^{2n}} d^\top p_i + 2\|d\|_\infty \left(q_i^\top A p_i - b_i^\top q_i \right), \end{aligned} \quad (65)$$

where we used the representation of optimal transport from paper [28] based on the definition of the ℓ_1 -norm.

Together with (65) and (63), we rewrite (62) as follows

$$\sum_{i=1}^m \mathcal{W}(\widehat{x}_{av}^N, y_i) - \min_{x \in \Delta_n} \sum_{i=1}^m \mathcal{W}(x, y_i) \leq \frac{(4 + 17\sqrt{2})L_\zeta}{N}$$

Equating this to $m\varepsilon$ we get the number of iteration for the Algorithm 11

$$N = \frac{(4 + 17\sqrt{2})L_\zeta}{m\varepsilon}. \quad (66)$$

Thus, the initial SPP scaled by $1/m$ will be solved with ε -precision. Next, we use Lemma 5.1 for (64) and get

$$\|\mathbf{W}\widehat{\mathbf{x}}^N\|_2 \leq \frac{17L_\zeta}{N} \cdot \frac{\lambda_{\min}^+(W)}{2\sqrt{2mn} \max_{i,j} C_{ij}}. \quad (67)$$

Then we use (66) and get

$$\|\mathbf{W}\widehat{\mathbf{x}}^N\|_2 \leq \frac{17\sqrt{m}\lambda_{\min}^+(W)\varepsilon}{2(4 + 17\sqrt{2})\sqrt{2n} \max_{i,j} C_{ij}}. \quad (68)$$

The complexity of one iteration of Algorithm 3 per node is $O(n^2)$ as the number of non-zero elements in matrix A is $2n^2$. Multiplying this by the number of iterations $N = (4 + 17\sqrt{2})L_\zeta/(m\varepsilon)$ we get

$$O(n^2 N) = O\left(n^2 \frac{L_\zeta}{m\varepsilon}\right) = O\left(\frac{n^2}{\varepsilon} \sqrt{n \log n} \max_{i,j} C_{ij} \chi\right),$$

where we used the definition of L_ζ from (31). □

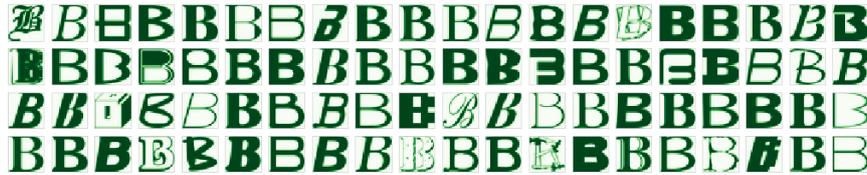
11. Algorithm and experimental data

11.1. Decentralized Mirror-Prox for Wasserstein Barycenters

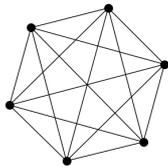
To present the algorithm we introduce soft-max function for an $x \in \mathbb{R}^n$

$$\text{Softmax}(x) = \frac{\exp(x)}{\sum_{i=1}^n (\exp(x))}$$

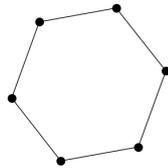
11.2. Letter 'B' in a variety of fonts from the notMNIST dataset



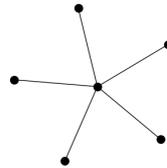
11.3. Network architectures



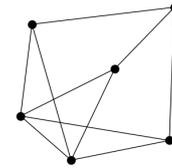
The complete graph



The cycle graph



The star graph



Erdős-Rényi random graph with probability of edge creation $p = 0.5$

12. Future work

First of all, the approach of this paper can be generalized to non-symmetric matrix W , but with Network compatibility and Kernel properties. This observation allows to improve the bounds by using proper weighting, see [21].

By using the standard restarts or regularization arguments, all the results of this paper have convex-concave or strongly convex-concave analogues. Unfortunately, optimality

Algorithm 3 Decentralized Mirror-Prox for Wasserstein Barycenters

Require: measures y_1, \dots, y_m , linearized cost matrix d , incidence matrix A , communication matrix W , number of iterations N , stepsize $\alpha = 1 / \left(16m\sqrt{2n \ln n} \max_{i,j} C_{ij}\chi \right)$

- 1: Initialize starting points $x^1 = \frac{1}{n}\mathbf{1}_n, p_1^1 = \dots = p_m^1 = \frac{1}{n^2}\mathbf{1}_{n^2}, q_1^1 = \dots = q_m^1 = \mathbf{0}_{2n}$
- 2: Initialize $\eta = \frac{\alpha mn}{\max_{i,j} C_{ij}^3 / \lambda_{\min}^2(W)}, \beta = 6\alpha m \log n, \theta = 2\alpha mn \max_{i,j} C_{ij}^2 / \lambda_{\min}^2(W)$
- 3: **for** $k = 1, 2, \dots, N - 1$ **do**
- 4: Each node $i = 1, \dots, m$ computes

$$\begin{aligned} u_i^{k+1} &= \text{Softmax} \left(\log(p_i^k) - \beta \left(d + 2 \max_{i,j} C_{ij} A^\top q_i^k \right) \right) \\ s_i^{k+1} &= \text{Softmax} \left(\log(x_i^k) + \beta \left([q_i^k]_{1\dots n} - \sum_{j=1}^m W_{ij} z_j^k \right) \right) \\ v_i^{k+1} &= q_i^k + \eta \left(A p_i^k - \begin{pmatrix} x_i^k \\ y_i \end{pmatrix} \right), \quad \text{project } v_i^{k+1} \text{ onto } [-1, 1]^{2n}. \\ \lambda_i^{k+1} &= z_i^k + \theta \sum_{j=1}^m W_{ij} x_j^k \end{aligned}$$

- 5: Each node $i = 1, \dots, m$ computes

$$\begin{aligned} p_i^{k+1} &= \text{Softmax} \left(\log(p_i^k) - \beta \left(d + 2 \max_{i,j} C_{ij} A^\top v_i^{k+1} \right) \right) \\ x_i^{k+1} &= \text{Softmax} \left(\log(x_i^k) + \beta \left([v_i^{k+1}]_{1\dots n} - \sum_{j=1}^m W_{ij} \lambda_j^{k+1} \right) \right) \\ q_i^{k+1} &= q_i^k + \eta \left(A u_i^{k+1} - \begin{pmatrix} s_i^{k+1} \\ y_i \end{pmatrix} \right), \quad \text{project } q_i^{k+1} \text{ onto } [-1, 1]^{2n}. \\ z_i^{k+1} &= z_i^k + \theta \sum_{j=1}^m W_{ij} s_j^{k+1} \end{aligned}$$

- 6: **end for**

Ensure: $\hat{\mathbf{x}}^N = (\hat{x}_1^N, \dots, \hat{x}_m^N)^\top$, where $\hat{x}_i^N = \frac{1}{N} \sum_{k=1}^N s_i^k$

w.r.t. ε take places only for the convex-concave case not for the strongly convex-concave one.² Our paper technique can be generalized to non-smooth problems by using another variant of sliding procedure [15, 23, 34]. By using batching technique, the results can be generalized to stochastic saddle-point problems [15, 23]. Instead of the smooth convex-concave saddle-point problem we can consider general sum-type saddle-point problems with common variables in more general form. For each group of common variable, we introduce corresponding communication network which includes the nodes correspond to the terms contain this variable. The bounds change according to the worth condition number of Laplacian matrices of these networks. Based on the lower bound, we expect that optimal algorithms for all the parameters for smooth (strongly) convex-concave saddle-point problems one can search as a combination Mirror-Prox with accelerated consensus algorithm, see [24] and references therein.

²The analysis developed in this paper also does not well fitted to the strongly convex-concave saddle-point problems with different constants of strong convexity and concavity, see the lower bound. In the non-distributed setup, this a popular direction of research [39]. For distributed setup, this is an open problem.

An interesting and open problem is generalizing the results of this paper to β -similar terms in the smooth convex-concave saddle-point problem [1, 56].

The results will probably change with the replacement of L by β .