# The First Optimal Algorithm for Smooth and Strongly-Convex-Strongly-Concave Minimax Optimization 

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#### Abstract

In this paper, we revisit the smooth and strongly-convex-strongly-concave minimax optimization problem. Zhang et al. (2021) and Ibrahim et al. (2020) established the lower bound $\Omega\left(\sqrt{k_{x} \kappa_{y}} \log \frac{1}{\epsilon}\right)$ on the number of gradient evaluations required to find an $\epsilon$-accurate solution, where $\kappa_{x}$ and $\kappa_{y}$ are condition numbers for the strong convexity and strong concavity assumptions. However, the existing state-of-the-art methods do not match this lower bound: algorithms of Lin et al. (2020) and Wang and Li (2020) have gradient evaluation complexity $\mathcal{O}\left(\sqrt{\kappa_{x} \kappa_{y}} \log ^{3} \frac{1}{\epsilon}\right)$ and $\mathcal{O}\left(\sqrt{\kappa_{x} \kappa_{y}} \log ^{3}\left(\kappa_{x} \kappa_{y}\right) \log \frac{1}{\epsilon}\right)$, respectively. We fix this fundamental issue by providing the first algorithm with $\mathcal{O}\left(\sqrt{\kappa_{x} \kappa_{y}} \log \frac{1}{\epsilon}\right)$ gradient evaluation complexity. We design our algorithm in three steps: (i) we reformulate the original problem as a minimization problem via the pointwise conjugate function; (ii) we apply a specific variant of the proximal point algorithm to the reformulated problem; (iii) we compute the proximal operator inexactly using the optimal algorithm for operator norm reduction in monotone inclusions.


## 1 Introduction

In this paper, we revisit the smooth and strongly-convex-strongly-concave minimax optimization problem of the form

$$
\begin{equation*}
\min _{x \in \mathbb{R}^{d_{x}}} \max _{y \in \mathbb{R}^{d_{y}}} r(x)+F(x, y)-g(y), \tag{1}
\end{equation*}
$$

where $F(x, y): \mathbb{R}^{d_{x}} \times \mathbb{R}^{d_{y}} \rightarrow \mathbb{R}$ is a continuously differentiable function, $r(x): \mathbb{R}^{d_{x}} \rightarrow \mathbb{R} \cup\{+\infty\}$ and $g(y): \mathbb{R}^{d_{y}} \rightarrow \mathbb{R} \cup\{+\infty\}$ are proper lower semi-continuous convex functions. Problem (1) has been actively studied in economics, game theory, statistics and computer science (Başar and Olsder 1998; Roughgarden, 2010; Von Neumann and Morgenstern. 1947, Facchinei and Pang, 2003 , Berger 2013). Recently, many applications of this problem appeared in machine learning, including adversarial training (Madry et al., 2017; Sinha et al., 2017), prediction and regression problems (Taskar et al., 2005, Xu et al., 2009), reinforcement learning (Du et al., 2017, Dai et al., 2018) and generative adversarial networks Arjovsky et al. (2017); Goodfellow et al. (2014).
In our paper, we focus on the case when function $f(x, y)$ is strongly convex in $x$ and strongly concave in $y$. There are several reasons to consider this function class. First, this setting is fundamental and studied by most existing works on minimax optimization ${ }^{3}$ Second, efficient algorithms initially

[^0]Table 1: Comparison of the state-of-the-art algorithms for solving smooth and strongly-convex-strongly-concave minimax problems in the number of gradient evaluations required to find an $\epsilon$-accurate solution (Definition 1).

| Reference | Gradient Complexity |
| :---: | :---: |
| Tseng (2000) | $\mathcal{O}\left(\max \left\{\kappa_{x}, \kappa_{y}\right\} \log \frac{1}{\epsilon}\right)$ |
| Nesterov and Scrimali (2006) |  |
| Gidel et al. (2018) |  |
| Alkousa et al. (2019) | $\mathcal{O}\left(\min \left\{\kappa_{x} \sqrt{\kappa_{y}}, \kappa_{y} \sqrt{\kappa_{x}}\right\} \log ^{2} \frac{1}{\epsilon}\right)$ |
| Lin et al. (2020) | $\mathcal{O}\left(\sqrt{\kappa_{x} \kappa_{y}} \log ^{3} \frac{1}{\epsilon}\right)$ |
| Wang and Li (2020) | $\mathcal{O}\left(\sqrt{\kappa_{x} \kappa_{y}} \log ^{3}\left(\kappa_{x} \kappa_{y}\right) \log \frac{1}{\epsilon}\right)$ |
| Algorithm 4 (This paper) | $\mathcal{O}\left(\sqrt{\kappa_{x} \kappa_{y}} \log \frac{1}{\epsilon}\right)$ |
| Lower Bound (Zhang et al., 2021; Ibrahim et al. 2020) | $\Omega\left(\sqrt{\kappa_{x} \kappa_{y}} \log \frac{1}{\epsilon}\right)$ |

developed for convex optimization often show state-of-the-art performance in non-convex applications (Kingma and Ba, 2014, Reddi et al., 2019; Duchi et al., 2011). Finally, we will further see that this fundamental setting is utterly understudied and lacks answers to even the most basic questions such as "What is the best possible algorithm for solving a problem in this setting?' ${ }^{4}$

### 1.1 Related Work

Until recently, the best-known gradient evaluation complexity of solving problem (1) was $\mathcal{O}\left(\max \left\{\kappa_{x}, \kappa_{y}\right\} \log \frac{1}{\epsilon}\right)$ (Tseng, 2000; Nesterov and Scrimali, 2006, Gidel et al., 2018), where $\kappa_{x}$ and $\kappa_{y}$ denote the condition numbers of functions $f(\cdot, y)$ and $f(x, \cdot)$, respectively. The first attempt to provide an algorithm with an "accelerated" convergence rate was the work of Alkousa et al. (2019). They provided an algorithm with $\mathcal{O}\left(\min \left\{\kappa_{x} \sqrt{\kappa_{y}}, \kappa_{y} \sqrt{\kappa_{x}}\right\} \log ^{2} \frac{1}{\epsilon}\right)$ gradient evaluation complexity. This result was subsequently improved up to $\mathcal{O}\left(\sqrt{\kappa_{x} \kappa_{y}} \log ^{3} \frac{1}{\epsilon}\right)$ by Lin et al. (2020) and $\mathcal{O}\left(\sqrt{\kappa_{x} \kappa_{y}} \log ^{3}\left(\kappa_{x} \kappa_{y}\right) \log \frac{1}{\epsilon}\right)$ by Wang and Li (2020). However, these results do not match the lower complexity bound $\Omega\left(\sqrt{\kappa_{x} \kappa_{y}} \log \frac{1}{\epsilon}\right)$ established by Zhang et al. (2021); Ibrahim et al. (2020). Hence, we have the following fundamental open problem:

Can we design an algorithm that achieves the lower gradient evaluation complexity bound in smooth and strongly-convex-strongly-concave minimax optimization?

It is worth mentioning that this open question was answered positively in the work of Kovalev et al. (2021) in the case of minimax problems with bilinear coupling, i.e., when $F(x, y)=p(x)+x$ A $y-$ $q(y)$, where $p(x)$ and $q(y)$ are smooth and strongly convex functions, and $\mathbf{A}$ is a $d_{x} \times d_{y}$ matrix. However, the algorithm provided in this work does not apply to the general minimax problem (1).

### 1.2 Main Contributions

We develop the first optimal algorithm for solving problem (1) in the smooth and strongly-convex-strongly-concave regime, which is the main contribution of this work. We split the algorithm development in three steps:
(i) In Section 3, we reformulate problem (1) as a particular minimization problem.
(ii) In Section 4, we develop a specific variant of the accelerated proximal point algorithm (Algorithm 2) which will be used as a baseline for the optimal algorithm construction.
(ii) In Section 5, we develop an optimal algorithm for operator norm reduction in monotone inclusion problems, which will be used for the proximal operator computation in Algorithm 2

[^1]In the final Section 6, we summarize these three steps by describing the optimal algorithm construction and showing that the complexity of the proposed algorithm matches the lower bound.

As mentioned before, in Section 5 , we develop an optimal algorithm for operator norm reduction in composite monotone inclusion problems of the form (22), which is the second main contribution of this work. To the best of our knowledge, there is only one optimal algorithm of Yoon and Ryu (2021), which works for Lipschitz-continuous operators only, i.e., when $B(u) \equiv 0$ in problem (22). In contrast to this, our algorithm works in the composite case with general maximally monotone operator $B(u)$.

## 2 Preliminaries

The following assumptions formalize the smoothness, strong convexity, and strong concavity properties of function $f(x, y)$.
Assumption 1. Function $F(x, y)$ is $\mu_{x}$-strongly convex in $x$, where $\mu_{x}>0$. That is, the following inequality holds for all $x_{1}, x_{2} \in \mathbb{R}^{d_{x}}, y \in \mathbb{R}^{d_{y}}$ :

$$
\begin{equation*}
F\left(x_{2}, y\right) \geq F\left(x_{1}, y\right)+\left\langle\nabla_{x} F\left(x_{1}, y\right), x_{2}-x_{1}\right\rangle+\left(\mu_{x} / 2\right)\left\|x_{2}-x_{1}\right\|^{2} \tag{2}
\end{equation*}
$$

Assumption 2. Function $F(x, y)$ is $\mu_{y}$-strongly concave in $y$, where $\mu_{y}>0$. That is, the following inequality holds for all $x \in \mathbb{R}^{d_{x}}, y_{1}, y_{2} \in \mathbb{R}^{d_{y}}$ :

$$
\begin{equation*}
F\left(x, y_{2}\right) \leq F\left(x, y_{1}\right)+\left\langle\nabla_{y} F\left(x, y_{1}\right), y_{2}-y_{1}\right\rangle-\left(\mu_{y} / 2\right)\left\|y_{2}-y_{1}\right\|^{2} . \tag{3}
\end{equation*}
$$

Assumption 3. Function $F(x, y)$ is L-smooth. That is, the following inequality holds for all $x_{1}, x_{2} \in \mathbb{R}^{d_{x}}, y_{1}, y_{2} \in \mathbb{R}^{d_{y}}:$

$$
\begin{equation*}
\left\|\nabla F\left(x_{1}, y_{1}\right)-\nabla F\left(x_{2}, y_{2}\right)\right\|^{2} \leq L^{2}\left(\left\|x_{1}-x_{2}\right\|^{2}+\left\|y_{1}-y_{2}\right\|^{2}\right) \tag{4}
\end{equation*}
$$

Under these assumptions, by $\kappa_{x}=\frac{L}{\mu_{x}}$ and $\kappa_{y}=\frac{L}{\mu_{x}}$, we denote the condition numbers of functions $F(\cdot, y)$ and $F(x, \cdot)$, respectively. The following assumption formalizes the properties of regularizers $r(x)$ and $g(y)$.
Assumption 4. Functions $r(x)$ and $g(y)$ are convex, lower semi-continuous and proper, i.e., there exist $\bar{x} \in \mathbb{R}^{d_{x}}, \bar{y} \in \mathbb{R}^{d_{y}}$ such that $r(\bar{x}), g(\bar{y})<+\infty$.
By $\left(x^{*}, y^{*}\right) \in \mathbb{R}^{d_{x}} \times \mathbb{R}^{d_{y}}$, we denote the solution of problem (1), which is characterized via the first-order optimality conditions

$$
\left\{\begin{align*}
-\nabla_{x} F\left(x^{*}, y^{*}\right) & \in \partial r\left(x^{*}\right)  \tag{5}\\
\nabla_{y} F\left(x^{*}, y^{*}\right) & \in \partial g\left(y^{*}\right)
\end{align*}\right.
$$

Note that there exists a unique solution to the problem due to the strong convexity and strong concavity assumptions (Assumptions 1 and 2). Hence, for any point $(x, y) \in R^{d_{x}} \times \mathbb{R}^{d_{y}}$, we can use squared distance to the solution $\left\|x-x^{*}\right\|^{2}+\left\|y-y^{*}\right\|^{2}$ as an optimality criterion. We formalize it through the following definition.
Definition 1. We call a pair of vectors $(x, y) \in \mathbb{R}^{d_{x}} \times \mathbb{R}^{d_{y}}$ an $\epsilon$-accurate solution of problem (1) for a given accuracy $\epsilon>0$ if it satisfies

$$
\begin{equation*}
\left\|x-x^{*}\right\|^{2}+\left\|y-y^{*}\right\|^{2} \leq \epsilon \tag{6}
\end{equation*}
$$

## 3 Step I: Reformulation via Pointwise Conjugate Function

In this section, we reformulate problem (1) as a particular convex minimization problem. This reformulation will be beneficial because minimization problems are typically easier to solve than minimax optimization problems.

### 3.1 Pointwise Conjugate Function

We start by introducing the pointwise conjugate function which will be the main component of our problem reformulation. Let function $\hat{F}(x, y): \mathbb{R}^{d_{x}} \times \mathbb{R}^{d_{y}} \rightarrow \mathbb{R}$ be defined as

$$
\begin{equation*}
\hat{F}(x, y)=F(x, y)-\frac{\mu_{x}}{2}\|x\|^{2}+\frac{\mu_{y}}{2}\|y\|^{2} \tag{7}
\end{equation*}
$$

One can observe that function $\hat{F}(x, y)$ is smooth, convex in $x$, and concave in $y$ due to Assumptions 1 to 3 . Now, the pointwise conjugate function $G(z, y): \mathbb{R}^{d_{x}} \times \mathbb{R}^{d_{y}} \rightarrow \mathbb{R}$ is defined as follows:

$$
\begin{equation*}
G(z, y)=\sup _{x \in \mathbb{R}^{d_{x}}}[\langle x, z\rangle-r(x)-\hat{F}(x, y)+g(y)] . \tag{8}
\end{equation*}
$$

One can observe that for fixed $y \in \mathbb{R}^{d_{y}}$, function $G(\cdot, y)$ is nothing else but the Fenchel conjugate ${ }^{5}$ of function $r(\cdot)+\hat{F}(\cdot, y)-g(y)$. Moreover, function $G(z, y)$ is defined as a pointwise supremum of a family of convex and lower semi-continuous functions $\left\{\varphi_{x}(z, y)=\langle x, z\rangle-r(x)-F(x, y)+g(y) \mid x \in \mathbb{R}^{d_{x}}\right\}$. Hence, $G(z, y)$ is also convex and lower semi-continuous function. The following lemma provides a characterization of the subdifferential of the pointwise conjugate function.
Lemma 1. Let $z, x \in \mathbb{R}^{d_{x}}$ and $y, w \in \mathbb{R}^{d_{y}}$ be arbitrary vectors that satisfy

$$
\begin{equation*}
z-\nabla_{x} \hat{F}(x, y) \in \partial r(x), w+\nabla_{y} \hat{F}(x, y) \in \partial g(y) \tag{9}
\end{equation*}
$$

Then, $G(z, y)=\langle z, x\rangle-r(x)-\hat{F}(x, y)+g(y)$ and $(x, w) \in \partial G(z, y)$.

### 3.2 Reformulation of the Minimax Optimization Problem

Now, we introduce the following minimization problem:

$$
\begin{equation*}
\min _{z \in \mathbb{R}^{d_{x}}, y \in \mathbb{R}^{d_{y}}}\left[P(z, y)=\frac{\mu_{x}^{-1}}{2}\|z\|^{2}+\frac{\mu_{y}}{2}\|y\|^{2}+G(z, y)\right] \tag{10}
\end{equation*}
$$

It turns out that this minimization problem can be seen as a reformulation of problem (1). This is justified by the following lemma.
Lemma 2. Problem 10 has a unique solution $\left(z^{*}, y^{*}\right) \in \mathbb{R}^{d_{x}} \times \mathbb{R}^{d_{y}}$, where

$$
\begin{equation*}
z^{*}=-\mu_{x} x^{*} \tag{11}
\end{equation*}
$$

and $\left(x^{*}, y^{*}\right)$ is the unique solution of problem (1).
Lemma 2 implies that if we find an approximate solution $(z, y) \in \mathbb{R}^{d_{x}} \times \mathbb{R}^{d_{y}}$ to problem (10), a pair of vectors $\left(-\mu_{x}^{-1} z, y\right) \in \mathbb{R}^{d_{x}} \times \mathbb{R}^{d_{y}}$ will be an approximate solution to the original minimax problem.
The idea of reformulating the minimax optimization problem as a minimization problem is not new and has been used in the state-of-the-art works of Lin et al. (2020); Wang and Li (2020); Alkousa et al. (2019). However, their reformulation is different from ours and has several disadvantages. In particular, it does not allow for building the optimal algorithm for solving problem (1). We provide a detailed discussion of this in the Appendix.

## 4 Step II: Accelerated Proximal Point Method

In this section, we develop the main algorithmic framework for solving problem (10), which is formalized as Algorithm 2 . We give the intuition behind the development of Algorithm 2 and provide its theoretical analysis. Further, in Section 6 we will use this algorithmic framework to develop the first optimal algorithm for solving main problem (1).

### 4.1 Nesterov Acceleration

It is well-known that Accelerated Gradient Method of Nesterov (1983, 2003) is the optimal algorithm for solving smooth (strongly-)convex minimization problems. Therefore, we could try to apply this method to solving problem (10), which is formalized as Algorithm 1 Note that we used the notation $\mu_{z}=\mu_{x}^{-1}$ in Algorithm 11, which is the strong convexity parameter of $P(z, y)$ in $z$. Unfortunately, function $P(z, y)$ can be non-smooth, and the gradient $\nabla P\left(z_{g}^{k}, y_{g}^{k}\right)$ can be undefined. It means that Algorithm 1 cannot be applied to problem (10).

[^2]```
Algorithm 1 Accelerated Gradient Method
    input: \(z^{0}=z_{f}^{0} \in \mathbb{R}^{d_{x}}, y^{0}=y_{f}^{0} \in \mathbb{R}^{d_{y}}\)
    parameters: \(\alpha \in(0,1], \eta_{z}, \eta_{y}, \theta_{z}, \theta_{y}>0, K \in\{1,2, \ldots\}\)
    for \(k=0,1,2, \ldots, K-1\) do
        \(\left(z_{g}^{k}, y_{g}^{k}\right)=\alpha\left(z^{k}, y^{k}\right)+(1-\alpha)\left(z_{f}^{k}, y_{f}^{k}\right)\)
        \(z_{f}^{k+1}=z_{g}^{k}-\theta_{z} \nabla_{z} P\left(z_{g}^{k}, y_{g}^{k}\right)\)
        \(y_{f}^{k+1}=y_{g}^{k}-\theta_{y} \nabla_{y} P\left(z_{g}^{k}, y_{g}^{k}\right)\)
        \(z^{k+1}=z^{k}+\eta_{z} \mu_{z}\left(z_{g}^{k}-z^{k}\right)+\eta_{z} \theta_{z}^{-1}\left(z_{f}^{k+1}-z_{g}^{k}\right)\)
        \(y^{k+1}=y^{k}+\eta_{y} \mu_{y}\left(y_{g}^{k}-y^{k}\right)+\eta_{y} \theta_{y}^{-1}\left(y_{f}^{k+1}-y_{g}^{k}\right)\)
    end for
    output: \(\left(z^{K}, y^{K}\right)\)
```

```
Algorithm 2 Accelerated Proximal Point Algorithm
    input: \(z^{0}=z_{f}^{0} \in \mathbb{R}^{d_{x}}, y^{0}=y_{f}^{0} \in \mathbb{R}^{d_{y}}\)
    parameters: \(\alpha \in(0,1], \eta_{z}, \eta_{y}, \theta_{y}>0, K \in\{1,2, \ldots\}\)
    for \(k=0,1,2, \ldots, K-1\) do
        \(\left(z_{g}^{k}, y_{g}^{k}\right)=\alpha\left(z^{k}, y^{k}\right)+(1-\alpha)\left(z_{f}^{k}, y_{f}^{k}\right)\)
        Find \(\left(x_{f}^{k+1}, y_{f}^{k+1}, z_{f}^{k+1}, w_{f}^{k+1}\right) \in \mathbb{R}^{d_{x}} \times \mathbb{R}^{d_{y}} \times \mathbb{R}^{d_{x}} \times \mathbb{R}^{d_{y}}\) that satisfy (18)
        \(z^{k+1}=z^{k}+\eta_{z} \mu_{x}^{-1}\left(z_{f}^{k+1}-z^{k}\right)-\eta_{z}\left(x_{f}^{k+1}+\mu_{x}^{-1} z_{f}^{k+1}\right)\)
        \(y^{k+1}=y^{k}+\eta_{y} \mu_{y}\left(y_{f}^{k+1}-y^{k}\right)-\eta_{y}\left(w_{f}^{k+1}+\mu_{y} y_{f}^{k+1}\right)\)
    end for
    output: \(\left(z^{K}, y^{K}\right)\)
```


### 4.2 Moreau-Yosida Regularization

In order to avoid the issues caused by the non-smoothness of function $P(z, y)$, we use the MoreauYosida regularization Moreau, 1962; Yosida 2012). Consider a function $P^{\theta_{z}, \theta_{y}}(z, y)$ defined in the following way:

$$
\begin{equation*}
P^{\theta_{z}, \theta_{y}}(z, y)=\min _{z^{+} \in \mathbb{R}^{d_{x}}, y^{+} \in \mathbb{R}^{d_{y}}} \frac{1}{2 \theta_{z}}\left\|z^{+}-z\right\|^{2}+\frac{1}{2 \theta_{y}}\left\|y^{+}-y\right\|^{2}+P\left(z^{+}, y^{+}\right) \tag{12}
\end{equation*}
$$

where $\theta_{z}, \theta_{y}>0$. Function $P^{\theta_{z}, \theta_{y}}(z, y)$ is called the Moreau envelope of function $P(z, y)$. The Moreau envelope has two crucial properties. First, it is a smooth function. Second, it has the same minimizers as function $P(z, y)$ :

$$
\begin{equation*}
\left(z^{*}, y^{*}\right)=\underset{z \in \mathbb{R}^{d_{x}}, y \in \mathbb{R}^{d_{y}}}{\arg \min P^{\theta_{z}, \theta_{y}}(z, y) . . . . . . . . .} \tag{13}
\end{equation*}
$$

The latter means that we could apply Accelerated Gradient Method to problem (13), which would give us an efficient algorithm for solving problem 10 . Further, we are going to construct such an algorithm.

### 4.3 Construction of the Algorithm

We start the construction of our algorithm by computing the gradient $\nabla P^{\theta_{z}, \theta_{y}}\left(z_{g}^{k}, y_{g}^{k}\right)$. The theory of the Moreau-Yosida regularization (Lemaréchal and Sagastizábal, 1997) suggests that the gradient of the Moreau envelope can be computed in the following way:

$$
\nabla P^{\theta_{z}, \theta_{y}}\left(z_{g}^{k}, y_{g}^{k}\right)=\left[\begin{array}{c}
\theta_{z}^{-1}\left(z_{g}^{k}-z_{f}^{k+1}\right)  \tag{14}\\
\theta_{y}^{-1}\left(y_{g}^{k}-y_{f}^{k+1}\right)
\end{array}\right]
$$

where $\left(z_{f}^{k+1}, y_{f}^{k+1}\right) \in \mathbb{R}^{d_{x}} \times \mathbb{R}^{d_{y}}$ is computed via the following auxiliary minimization problem:

$$
\begin{equation*}
\left(z_{f}^{k+1}, y_{f}^{k+1}\right)=\underset{z \in \mathbb{R}^{d_{x}}, y \in \mathbb{R}^{d_{y}}}{\arg \min } \frac{1}{2 \theta_{z}}\left\|z-z_{g}^{k}\right\|^{2}+\frac{1}{2 \theta_{y}}\left\|y-y_{g}^{k}\right\|^{2}+P(z, y) \tag{15}
\end{equation*}
$$

Further, we choose parameter $\theta_{z}=\mu_{z}^{-1}=\mu_{x}$ and write the first-order optimality conditions for this problem using the definition of function $P(z, y)$ :

$$
\left[\begin{array}{c}
\mu_{x}^{-1}\left(z_{f}^{k+1}-z_{g}^{k}\right)+\mu_{x}^{-1} z_{f}^{k+1}  \tag{16}\\
\theta_{y}^{-1}\left(y_{f}^{k+1}-y_{g}^{k}\right)+\mu_{y} y_{f}^{k+1}
\end{array}\right] \in-\partial G\left(z_{f}^{k+1}, y_{f}^{k+1}\right)
$$

The latter condition involves the subdifferential $\partial G(z, y)$. Hence, we can rewrite this condition using Lemma 1, which provides the characterization of $\partial G(z, y)^{6}$.

$$
\begin{align*}
z_{f}^{k+1}-\nabla_{x} \hat{F}\left(x_{f}^{k+1}, y_{f}^{k+1}\right) \in \partial r\left(x_{f}^{k+1}\right), & x_{f}^{k+1}+\mu_{x}^{-1}\left(z_{f}^{k+1}-z_{g}^{k}\right)+\mu_{x}^{-1} z_{f}^{k+1}=0,  \tag{17}\\
w_{f}^{k+1}+\nabla_{y} \hat{F}\left(x_{f}^{k+1}, y_{f}^{k+1}\right) \in \partial g\left(y_{f}^{k+1}\right), & w_{f}^{k+1}+\theta_{y}^{-1}\left(y_{f}^{k+1}-y_{g}^{k}\right)+\mu_{y} y_{f}^{k+1}=0
\end{align*}
$$

where $x_{f}^{k+1} \in \mathbb{R}^{d_{x}}$ and $w_{f}^{k+1} \in \mathbb{R}^{d_{y}}$ are auxiliary vectors. From 17] we get

$$
\begin{aligned}
\mu_{x}^{-1}\left(z_{f}^{k+1}-z_{g}^{k}\right) & =-\left(x_{f}^{k+1}+\mu_{x}^{-1} z_{f}^{k+1}\right) \\
\theta_{y}^{-1}\left(y_{f}^{k+1}-y_{g}^{k}\right) & =-\left(w_{f}^{k+1}+\mu_{y} y_{f}^{k+1}\right)
\end{aligned}
$$

which we plug into lines 7 and 8 of Algorithm 1
Finally, we replace the computation of $\left(z_{f}^{k+1}, y_{f}^{k+1}\right)$ on lines 5 and 6 of Algorithm 1 using condition (17). It turns out that we can use the following relaxed version of condition (17) without hurting the convergence properties of the resulting algorithm:

$$
\left\{\begin{array}{l}
z_{f}^{k+1}-\nabla_{x} \hat{F}\left(x_{f}^{k+1}, y_{f}^{k+1}\right) \in \partial r\left(x_{f}^{k+1}\right)  \tag{18}\\
w_{f}^{k+1}+\nabla_{y} \hat{F}\left(x_{f}^{k+1}, y_{f}^{k+1}\right) \in \partial g\left(y_{f}^{k+1}\right) \\
\frac{8}{\mu_{x}}\left\|\Delta_{x}^{k}\right\|^{2}+\theta_{y}\left\|\Delta_{y}^{k}\right\|^{2} \leq \frac{\mu_{x}}{8}\left\|x_{f}^{k+1}+\mu_{x}^{-1} z_{g}^{k}\right\|^{2}+\theta_{y}^{-1}\left\|y_{f}^{k+1}-y_{g}^{k}\right\|^{2}
\end{array}\right.
$$

where $\Delta_{x}^{k}$ and $\Delta_{y}^{k}$ are defined as follows:

$$
\left\{\begin{array}{l}
\Delta_{x}^{k}=z_{f}^{k+1}+\frac{\mu_{x}}{2}\left(x_{f}^{k+1}-\mu_{x}^{-1} z_{g}^{k}\right)  \tag{19}\\
\Delta_{y}^{k}=w_{f}^{k+1}+\mu_{y} y_{f}^{k+1}+\theta_{y}^{-1}\left(y_{f}^{k+1}-y_{g}^{k}\right)
\end{array}\right.
$$

### 4.4 Convergence of the Algorithm

After applying all the modifications mentioned above to Algorithm 1 , we obtain Algorithm 2, Theorem 1] provides the iteration complexity of Algorithm 2. The proof of Theorem 1 can be found in the Appendix.
Theorem 1. Let $\eta_{z}, \eta_{y}$ be defined as

$$
\begin{equation*}
\eta_{z}=\mu_{x} / 2, \quad \eta_{y}=\min \left\{1 /\left(2 \mu_{y}\right), \theta_{y} /(2 \alpha)\right\} . \tag{20}
\end{equation*}
$$

Then, to find an $\epsilon$-accurate solution of problem (1), Algorithm 2 requires the following number of iterations:

$$
\begin{equation*}
K=\mathcal{O}\left(\max \left\{\frac{1}{\alpha}, \frac{\alpha}{\theta_{y} \mu_{y}}\right\} \log \frac{1}{\epsilon}\right) . \tag{21}
\end{equation*}
$$

In this case, the $\epsilon$-accurate solution will be given as $\left(-\mu_{x}^{-1} z^{K}, y^{K}\right)$, where $\left(z^{K}, y^{K}\right)$ is the output of Algorithm 2

Unfortunately, Algorithm 2 cannot be applied to solving problem (1) in its current form because it requires finding vectors $\left(x_{f}^{k+1}, y_{f}^{k+1}, z_{f}^{k+1}, w_{f}^{k+1}\right)$ that satisfy condition 18 on line 5 at each iteration. Further, we will show that finding these vectors can be seen as finding an approximate solution to a particular monotone inclusion problem. In Section 5, we will provide an optimal algorithm for solving such monotone inclusions. In Section 6, we will show how to combine this algorithm with Algorithm 2 and obtain the first optimal algorithm for solving main problem (1).

```
Algorithm 3 Extra Anchored Gradient for Monotone Inclusions
    input: \(u^{-1} \in \mathbb{R}^{d}\)
    parameters: \(\lambda>0, T \in\{1,2, \ldots\},\left\{\beta_{t}\right\}_{t=0}^{T-1} \subset(0,1)\)
    \(u^{0}=\mathrm{J}_{\lambda B}\left(u^{-1}-\lambda A\left(u^{-1}\right)\right)\)
    \(a^{0}=A\left(u^{0}\right)\)
    \(b^{0}=\frac{1}{\lambda}\left(u^{-1}-\lambda A\left(u^{-1}\right)-u^{0}\right) \quad \triangleright b^{0} \in B\left(u^{0}\right)\)
    for \(t=0,1,2 \ldots, T-1\) do
        \(u^{t+1 / 2}=u^{t}+\beta_{t}\left(u^{0}-u^{t}\right)-\lambda\left(a^{t}+b^{t}\right)\)
        \(u^{t+1}=\mathrm{J}_{\lambda B}\left(u^{t}+\beta_{t}\left(u^{0}-u^{t}\right)-\lambda A\left(u^{t+1 / 2}\right)\right)\)
        \(a^{t+1}=A\left(u^{t+1}\right)\)
        \(b^{t+1}=\frac{1}{\lambda}\left(u^{t}+\beta_{t}\left(u^{0}-u^{t}\right)-\lambda A\left(u^{t+1 / 2}\right)-u^{t+1}\right) \quad \triangleright b^{t+1} \in B\left(u^{t+1}\right)\)
    end for
    output: \(\left(u^{T}, a^{T}+b^{T}\right) \quad \triangleright b^{T} \in B\left(u^{T}\right)\)
```


## 5 Step III: Operator Norm Reduction in Monotone Inclusions

In this section, we consider the following monotone inclusion problem:

$$
\begin{equation*}
\text { find } u^{*} \in \mathbb{R}^{d} \quad \text { such that } \quad 0 \in A\left(u^{*}\right)+B\left(u^{*}\right) \tag{22}
\end{equation*}
$$

where $A(u), B(u): \mathbb{R}^{d} \rightrightarrows \mathbb{R}^{d}$ are maximally monotone mappings. We are interested in the case when $A(u)$ is single-valued and Lipschitz continuous. The properties of operators $A(u)$ and $B(u)$ are formalized through the following assumptions.
Assumption 5. Mapping $A(u): \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ is single-valued, $M$-Lipschitz and monotone. That is, for all $u_{1}, u_{2} \in \mathbb{R}^{d},\left\langle A\left(u_{1}\right)-A\left(u_{2}\right), u_{1}-u_{2}\right\rangle \geq 0$ and $\left\|A\left(u_{1}\right)-A\left(u_{2}\right)\right\| \leq M\left\|u_{1}-u_{2}\right\|$.
Assumption 6. Mapping $B(u): \mathbb{R}^{d} \rightrightarrows \mathbb{R}^{d}$ is maximally monotone and possibly multivalued. That is, mapping $B(u)$ satisfies the following conditions:

1. $B(u)$ is monotone, i.e., for all $u_{1}, u_{2} \in \operatorname{dom} B, b_{1} \in B\left(u_{1}\right), b_{2} \in B\left(u_{2}\right)$ the following inequality holds: $\left\langle u_{1}-u_{2}, b_{1}-b_{2}\right\rangle \geq 0$, where $\operatorname{dom} B=\left\{u \in \mathbb{R}^{d} \mid B(u) \neq \emptyset\right\}$.
2. The graph $\operatorname{gph} B=\left\{(u, b) \in \mathbb{R}^{d} \times \mathbb{R}^{d} \mid b \in B(u)\right\}$ is not properly contained in the graph of any other monotone mapping on $\mathbb{R}^{d}$.

Note that mapping $A(u)$ is also maximally monotone because it is monotone and continuous Rockafellar and Wets, 2009, Example 12.7). Further, we will use an operator $\mathrm{J}_{\lambda B}(u): \mathbb{R}^{d} \rightrightarrows \mathbb{R}^{d}$ which is defined as

$$
\begin{equation*}
u^{+} \in \mathrm{J}_{\lambda B}(u) \quad \text { if and only if } \quad \lambda^{-1}\left(u-u^{+}\right) \in B\left(u^{+}\right) \tag{23}
\end{equation*}
$$

where $\lambda>0$. This mapping is called the resolvent of mapping $B(u)$. The maximal monotonicity of $B(u)$ implies that the resolvent $\mathrm{J}_{\lambda B}(u)$ is single-valued for all $u \in \mathbb{R}^{d}$ (Rockafellar and Wets, 2009, Theorem 12.12).
Now, we are ready to present Algorithm 3 for solving monotone inclusion problem (22). The design of our algorithm is based on the Extra Anchored Gradient Algorithm of Yoon and Ryu (2021). The critical difference between the algorithm of Yoon and Ryu (2021) and Algorithm 3 is that the algorithm of Yoon and Ryu (2021) can be applied to problem (22) in the case $B(u) \equiv\{0\}$ only. Therefore, our Algorithm 3 can be seen as an extension of the algorithm of Yoon and Ryu (2021) for general monotone inclusion problems of the form (22).
The following theorem provides the convergence guarantees for Algorithm 3 The proof of the theorem can be found in the Appendix.
Theorem 2. Assume that there exists at least a single solution $u^{*}$ to problem (22). Let $\beta_{t}$ be defined as follows

$$
\begin{equation*}
\beta_{t}=2 /(t+3) \tag{24}
\end{equation*}
$$

[^3]```
Algorithm 4 FOAM: The First Optimal Algorithm for Minimax Optimization
    input: \(z^{0}=z_{f}^{0} \in \mathbb{R}^{d_{x}}, y^{0}=y_{f}^{0} \in \mathbb{R}^{d_{y}}\)
    parameters: \(\alpha \in(0,1], \eta_{z}, \eta_{y}, \theta_{y}>0,\left\{\beta_{t}\right\}_{t=0}^{\infty} \subset(0,1), \lambda, \gamma_{x}, \gamma_{y}>0, K \in\{1,2, \ldots\}\)
    for \(k=0,1,2, \ldots, K-1\) do
        \(\left(z_{g}^{k}, y_{g}^{k}\right)=\alpha\left(z^{k}, y^{k}\right)+(1-\alpha)\left(z_{f}^{k}, y_{f}^{k}\right)\)
        \(\left(x^{k,-1}, y^{k,-1}\right)=\left(-\mu_{x}^{-1} z_{g}^{k}, y_{g}^{k}\right)\)
        \(x^{k, 0}=\operatorname{prox}_{\gamma_{x} \lambda r(\cdot)}\left(x^{k,-1}-\gamma_{x} \lambda a_{x}^{k}\left(x^{k,-1}, y^{k,-1}\right)\right)\)
        \(y^{k, 0}=\operatorname{prox}_{\gamma_{y} \lambda g(\cdot)}\left(y^{k,-1}-\gamma_{y} \lambda a_{y}^{k}\left(x^{k,-1}, y^{k,-1}\right)\right)\)
        \(b_{x}^{k, 0}=\frac{1}{\gamma_{x} \lambda}\left(x^{k,-1}-\gamma_{x} \lambda a_{x}^{k}\left(x^{k,-1}, y^{k,-1}\right)-x^{k, 0}\right)\)
        \(b_{y}^{k, 0}=\frac{1}{\gamma_{y} \lambda}\left(y^{k,-1}-\gamma_{y} \lambda a_{y}^{k}\left(x^{k,-1}, y^{k,-1}\right)-y^{k, 0}\right)\)
        \(t=0\)
        while condition (31) is not satisfied do
            \(x^{k, t+1 / 2}=x^{k, t}+\beta_{t}\left(x^{k, 0}-x^{k, t}\right)-\gamma_{x} \lambda\left(a_{x}^{k}\left(x^{k, t}, y^{k, t}\right)+b_{x}^{k, t}\right)\)
            \(y^{k, t+1 / 2}=y^{k, t}+\beta_{t}\left(y^{k, 0}-y^{k, t}\right)-\gamma_{y} \lambda\left(a_{y}^{k}\left(x^{k, t}, y^{k, t}\right)+b_{y}^{k, t}\right)\)
            \(x^{k, t+1}=\operatorname{prox}_{\gamma_{x} \lambda r(\cdot)}\left(x^{k, t}+\beta_{t}\left(x^{k, 0}-x^{k, t}\right)-\gamma_{x} \lambda a_{x}^{k}\left(x^{k, t+1 / 2}, y^{k, t+1 / 2}\right)\right)\)
            \(y^{k, t+1}=\operatorname{prox}_{\gamma_{y} \lambda g(\cdot)}\left(y^{k, t}+\beta_{t}\left(y^{k, 0}-y^{k, t}\right)-\gamma_{y} \lambda a_{y}^{k}\left(x^{k, t+1 / 2}, y^{k, t+1 / 2}\right)\right)\)
            \(b_{x}^{k, t+1}=\frac{1}{\gamma_{x} \lambda}\left(x^{k, t}+\beta_{t}\left(x^{k, 0}-x^{k, t}\right)-\gamma_{x} \lambda a_{x}^{k}\left(x^{k, t+1 / 2}, y^{k, t+1 / 2}\right)-x^{k, t+1}\right)\)
            \(b_{y}^{k, t+1}=\frac{1}{\gamma_{y} \lambda}\left(y^{k, t}+\beta_{t}\left(y^{k, 0}-y^{k, t}\right)-\gamma_{y} \lambda a_{y}^{k}\left(x^{k, t+1 / 2}, y^{k, t+1 / 2}\right)-y^{k, t+1}\right)\)
            \(t=t+1\)
        end while
        \(t^{k}=t\)
        \(\left(x_{f}^{k+1}, y_{f}^{k+1}\right)=\left(x^{k, t^{k}}, y^{k, t^{k}}\right)\)
        \(\left(z_{f}^{k+1}, w_{f}^{k+1}\right)=\left(\nabla_{x} \hat{F}\left(x_{f}^{k+1}, y_{f}^{k+1}\right)+b_{x}^{k, t^{k}},-\nabla_{y} \hat{F}\left(x_{f}^{k+1}, y_{f}^{k+1}\right)+b_{y}^{k, t^{k}}\right)\)
        \(z^{k+1}=z^{k}+\eta_{z} \mu_{x}^{-1}\left(z_{f}^{k+1}-z^{k}\right)-\eta_{z}\left(x_{f}^{k+1}+\mu_{x}^{-1} z_{f}^{k+1}\right)\)
        \(y^{k+1}=y^{k}+\eta_{y} \mu_{y}\left(y_{f}^{k+1}-y^{k}\right)-\eta_{y}\left(w_{f}^{k+1}+\mu_{y} y_{f}^{k+1}\right)\)
    end for
    output: \(\left(-\mu_{x}^{-1} z^{K}, y^{K}\right)\)
```

Let $\lambda$ be defined as

$$
\begin{equation*}
\lambda=1 /(\sqrt{5} M) \tag{25}
\end{equation*}
$$

Then, the following inequality holds

$$
\begin{equation*}
\left\|a^{T}+b^{T}\right\|^{2} \leq \frac{288 M^{2}}{(T+1)^{2}}\left\|u^{-1}-u^{*}\right\|^{2} \tag{26}
\end{equation*}
$$

where $a^{T}=A\left(u^{T}\right)$ and $b^{T} \in B\left(u^{T}\right)$.

## 6 Final Step: The First Optimal Algorithm for Minimax Optimization

In this section, we construct the first optimal algorithm for solving main problem (1). In order to do this, we use Algorithm 3 to compute vectors $\left(x_{f}^{k+1}, y_{f}^{k+1}, z_{f}^{k+1}, w_{f}^{k+1}\right)$ on line 5 of Algorithm 2 Further, we describe the construction of our algorithm in detail.

### 6.1 Construction of the Algorithm

As mentioned in Section 4, Algorithm 2 cannot be applied to solving problem (1) in its current form because it requires finding the vectors satisfying condition (18) on line 5 at each iteration. Further, we will show how to do this using Algorithm 3 . Let $\mathbb{R}^{d}=\mathbb{R}^{d_{x}} \times \mathbb{R}^{d_{y}}$. For each $k \in\{0,1,2, \ldots\}$ consider operators $A^{k}(u): \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ and $B(u): \mathbb{R}^{d} \rightrightarrows \mathbb{R}^{d}$ defined as follows:

$$
A^{k}(u)=\left[\begin{array}{c}
\sqrt{\gamma_{x}} a_{x}^{k}(x, y)  \tag{27}\\
\sqrt{\gamma_{y}} a_{y}^{k}(x, y)
\end{array}\right], \quad B(u)=\left\{\left.\left[\begin{array}{c}
\sqrt{\gamma_{x}} b_{x} \\
\sqrt{\gamma_{y}} b_{y}
\end{array}\right] \right\rvert\, b_{x} \in \partial r(x), b_{y} \in \partial g(y)\right\}
$$

where $\gamma_{x}, \gamma_{y}>0$ are parameters, variable $u \in \mathbb{R}^{d}$ is defined as

$$
\begin{equation*}
u=\left(\gamma_{x}^{-1 / 2} x, \gamma_{y}^{-1 / 2} y\right), \quad \text { where } \quad(x, y) \in \mathbb{R}^{d_{x}} \times \mathbb{R}^{d_{y}} \tag{28}
\end{equation*}
$$

and operators $a_{x}^{k}(x, y): \mathbb{R}^{d} \rightarrow \mathbb{R}^{d_{x}}$ and $a_{y}^{k}(x, y): \mathbb{R}^{d} \rightarrow \mathbb{R}^{d_{y}}$ are defined as

$$
\begin{align*}
& a_{x}^{k}(x, y)=\nabla_{x} \hat{F}(x, y)+\frac{\mu_{x}}{2}\left(x-\mu_{x}^{-1} z_{g}^{k}\right)  \tag{29}\\
& a_{y}^{k}(x, y)=-\nabla_{y} \hat{F}(x, y)+\mu_{y} y+\theta_{y}^{-1}\left(y-y_{g}^{k}\right)
\end{align*}
$$

One can observe that operators $A^{k}(u)$ and $B(u)$ satisfy Assumptions 5 and 6 . This is justified by the following lemma.
Lemma 3. Operator $A^{k}(u)$, defined by (27), is monotone and $M$-Lipschitz, where $M$ is given as

$$
\begin{equation*}
M=2 \max \left\{\gamma_{x} L, \gamma_{y}\left(L+\theta_{y}^{-1}\right)\right\} . \tag{30}
\end{equation*}
$$

Operator $B(u)$, defined by (27), is maximally monotone.
Now, we are ready to construct the first optimal algorithm for solving main problem (1) which is formalized as Algorithm 4 In order to do this, we use Algorithm 3 to perform the computations on line 5 of Algorithm 2 . Consider the $k$-th iteration of Algorithm 2 and replace line 5 of Algorithm 2 with the lines of Algorithm 3 using the notation $u^{t}=\left(\gamma_{x}^{-1 / 2} x^{k, t}, \gamma_{y}^{-1 / 2} y^{k, t}\right)$ for $t \in\{-1,0,1,2, \ldots\}$.

In addition, we replace the for-loop of Algorithm 3 with the while-loop that iterates until the following condition is satisfied (see line 11 of Algorithm 4):

$$
\begin{align*}
\gamma_{x} \| a_{x}^{k}\left(x^{k, t}, y^{k, t}\right)+ & b_{x}^{k, t}\left\|^{2}+\gamma_{y}\right\| a_{y}^{k}\left(x^{k, t}, y^{k, t}\right)+b_{y}^{k, t} \|^{2} \leq \\
& \leq \gamma_{x}^{-1}\left\|x^{k, t}-x^{k,-1}\right\|^{2}+\gamma_{y}^{-1}\left\|y^{k, t}-y^{k,-1}\right\|^{2} . \tag{31}
\end{align*}
$$

We also set the initial iterates to $x^{k,-1}=-\mu_{x}^{-1} z_{g}^{k}$ and $y^{k,-1}=y_{g}^{k}$ on line 5 of Algorithm 4, and use the output of the inner while-loop to compute vectors ( $x_{f}^{k+1}, y_{f}^{k+1}, z_{f}^{k+1}, w_{f}^{k+1}$ ) on lines 21 and 22 of Algorithm 4 Now, if we define parameters $\gamma_{x}, \gamma_{y}$ in the following way:

$$
\begin{equation*}
\gamma_{x}=8 \mu_{x}^{-1}, \quad \gamma_{y}=\theta_{y} \tag{32}
\end{equation*}
$$

then condition (31) on line 11 of Algorithm 4 becomes equivalent to condition (18) on line 5 of Algorithm 2 Hence, vectors $\left(x_{f}^{k+1}, y_{f}^{k+1}, z_{f}^{k+1}, w_{f}^{k+1}\right.$ ) computed on lines 21 and 22 of Algorithm 4 satisfy condition (18), which implies that Algorithm4is a special case of Algorithm 2

### 6.2 Complexity of the Algorithm

It remains to establish the gradient evaluation complexity of Algorithm4. First, we need to estimate the number of iterations performed by the inner while-loop of Algorithm 4, which is equal to $t^{k}$ defined on line 20 of Algorithm 4 Recall that the inner while-loop was constructed out of the lines of Algorithm 3 Hence, we can use Theorem 2 to provide an upper bound on $t^{k}$. This is done by the following lemma.
Lemma 4. Assume the following choice of the parameters of Algorithm 4. stepsize $\lambda$ is defined by (25), parameter $M$ is defined by (30), sequence $\left\{\beta_{t}\right\}_{t=0}^{\infty}$ is defined by (24), parameters $\gamma_{x}$ and $\gamma_{y}$ are defined by (32). Then, $t^{k} \leq T$, where $T$ is given as

$$
\begin{equation*}
T=\left\lceil 48 \sqrt{2} \max \left\{8 L / \mu_{x}, 1+\theta_{y} L\right\}\right\rceil-1 \tag{33}
\end{equation*}
$$

Now, we are ready to provide the final gradient complexity of Algorithm4 It is done by the following theorem.
Theorem 3. Let parameters of Algorithm 4 be defined as follows: $\alpha=\min \left\{1, \sqrt{\theta_{y} \mu_{y}}\right\}, \theta_{y}=8 \mu_{x}^{-1}$, $\lambda=\left(2 \sqrt{5}\left(1+8 L / \mu_{x}\right)\right)^{-1}$, stepsizes $\eta_{z}$ and $\eta_{y}$ are defined by 20 , parameters $\gamma_{x}$ and $\gamma_{y}$ are defined by (32), parameters $\left\{\beta_{t}\right\}_{t=0}^{\infty}$ are defined by (24). Then, to find an $\epsilon$-accurate solution of problem (1), Algorithm 4 requires the following number of gradient evaluations:

$$
\begin{equation*}
\mathcal{O}\left(\max \left\{\frac{L}{\mu_{x}}, \frac{L}{\sqrt{\mu_{x} \mu_{y}}}\right\} \log \frac{1}{\epsilon}\right) . \tag{34}
\end{equation*}
$$

Corollary 1. Without loss of generality we can assume $\mu_{x} \geq \mu_{y}$, otherwise we just swap variables $x$ and $y$ in problem (1). Hence, Algorithm 4 has the following gradient evaluation complexity:

$$
\begin{equation*}
\mathcal{O}\left(\frac{L}{\sqrt{\mu_{x} \mu_{y}}} \log \frac{1}{\epsilon}\right) . \tag{35}
\end{equation*}
$$

## References

Alkousa, M., Dvinskikh, D., Stonyakin, F., Gasnikov, A., and Kovalev, D. (2019). Accelerated methods for composite non-bilinear saddle point problem. arXiv preprint arXiv:1906.03620.

Arjovsky, M., Chintala, S., and Bottou, L. (2017). Wasserstein generative adversarial networks. In International conference on machine learning, pages 214-223. PMLR.

Başar, T. and Olsder, G. J. (1998). Dynamic noncooperative game theory. SIAM.
Berger, J. O. (2013). Statistical decision theory and Bayesian analysis. Springer Science \& Business Media.

Dai, B., Shaw, A., Li, L., Xiao, L., He, N., Liu, Z., Chen, J., and Song, L. (2018). Sbeed: Convergent reinforcement learning with nonlinear function approximation. In International Conference on Machine Learning, pages 1125-1134. PMLR.

Du, S. S., Chen, J., Li, L., Xiao, L., and Zhou, D. (2017). Stochastic variance reduction methods for policy evaluation. In International Conference on Machine Learning, pages 1049-1058. PMLR.

Duchi, J., Hazan, E., and Singer, Y. (2011). Adaptive subgradient methods for online learning and stochastic optimization. Journal of machine learning research, 12(7).

Facchinei, F. and Pang, J.-S. (2003). Finite-dimensional variational inequalities and complementarity problems. Springer.

Gidel, G., Berard, H., Vignoud, G., Vincent, P., and Lacoste-Julien, S. (2018). A variational inequality perspective on generative adversarial networks. arXiv preprint arXiv:1802.10551.

Goodfellow, I., Pouget-Abadie, J., Mirza, M., Xu, B., Warde-Farley, D., Ozair, S., Courville, A., and Bengio, Y. (2014). Generative adversarial nets. Advances in neural information processing systems, 27.

Ibrahim, A., Azizian, W., Gidel, G., and Mitliagkas, I. (2020). Linear lower bounds and conditioning of differentiable games. In International conference on machine learning, pages 4583-4593. PMLR.

Kingma, D. P. and Ba, J. (2014). Adam: A method for stochastic optimization. arXiv preprint arXiv:1412.6980.

Kovalev, D., Gasnikov, A., and Richtárik, P. (2021). Accelerated primal-dual gradient method for smooth and convex-concave saddle-point problems with bilinear coupling. arXiv preprint arXiv:2112.15199.

Lemaréchal, C. and Sagastizábal, C. (1997). Practical aspects of the moreau-yosida regularization: Theoretical preliminaries. SIAM journal on optimization, 7(2):367-385.

Lin, T., Jin, C., and Jordan, M. I. (2020). Near-optimal algorithms for minimax optimization. In Conference on Learning Theory, pages 2738-2779. PMLR.

Madry, A., Makelov, A., Schmidt, L., Tsipras, D., and Vladu, A. (2017). Towards deep learning models resistant to adversarial attacks. arXiv preprint arXiv:1706.06083.

Moreau, J. J. (1962). Fonctions convexes duales et points proximaux dans un espace hilbertien. Comptes rendus hebdomadaires des séances de l'Académie des sciences, 255:2897-2899.

Nesterov, Y. (2003). Introductory lectures on convex optimization: A basic course, volume 87. Springer Science \& Business Media.

Nesterov, Y. and Scrimali, L. (2006). Solving strongly monotone variational and quasi-variational inequalities.

Nesterov, Y. E. (1983). A method for solving the convex programming problem with convergence rate o (1/k^2). In Dokl. akad. nauk Sssr, volume 269, pages 543-547.

Reddi, S. J., Kale, S., and Kumar, S. (2019). On the convergence of adam and beyond. arXiv preprint arXiv:1904.09237.

Rockafellar, R. T. and Wets, R. J.-B. (2009). Variational analysis, volume 317. Springer Science \& Business Media.

Roughgarden, T. (2010). Algorithmic game theory. Communications of the ACM, 53(7):78-86.
Sinha, A., Namkoong, H., and Duchi, J. (2017). Certifiable distributional robustness with principled adversarial training. arXiv preprint arXiv:1710.10571, 2.

Taskar, B., Lacoste-Julien, S., and Jordan, M. (2005). Structured prediction via the extragradient method. Advances in neural information processing systems, 18.

Tseng, P. (2000). A modified forward-backward splitting method for maximal monotone mappings. SIAM Journal on Control and Optimization, 38(2):431-446.

Von Neumann, J. and Morgenstern, O. (1947). Theory of games and economic behavior, 2nd rev.
Wang, Y. and Li, J. (2020). Improved algorithms for convex-concave minimax optimization. Advances in Neural Information Processing Systems, 33:4800-4810.

Xu, H., Caramanis, C., and Mannor, S. (2009). Robustness and regularization of support vector machines. Journal of machine learning research, 10(7).

Yoon, T. and Ryu, E. K. (2021). Accelerated algorithms for smooth convex-concave minimax problems with o $\left(1 / \mathrm{k}^{\wedge} 2\right)$ rate on squared gradient norm. In International Conference on Machine Learning, pages 12098-12109. PMLR.

Yosida, K. (2012). Functional analysis. Springer Science \& Business Media.
Zhang, J., Hong, M., and Zhang, S. (2021). On lower iteration complexity bounds for the convex concave saddle point problems. Mathematical Programming.

## Appendix

## A Proof of Lemma 1

From the definition of $G(z, y)$ it follows that

$$
\begin{equation*}
G(z, y) \geq\langle z, x\rangle-r(x)-\hat{F}(x, y)+g(y) \tag{36}
\end{equation*}
$$

Using $z-\nabla_{x} \hat{F}(x, y) \in \partial r(x)$, for arbitrary $\bar{x} \in \mathbb{R}^{d_{x}}$ we get

$$
\begin{aligned}
r(\bar{x}) & \geq r(x)+\left\langle z-\nabla_{x} \hat{F}(x, y), \bar{x}-x\right\rangle \\
& \geq r(x)+\langle z, \bar{x}-x\rangle+\hat{F}(x, y)-\hat{F}(\bar{x}, y)
\end{aligned}
$$

where we used the convexity of $\hat{F}(x, y)$ in $x$ in the last inequality. After rearranging we get

$$
\langle z, x\rangle-r(x)-\hat{F}(x, y) \geq\langle z, \bar{x}\rangle-r(\bar{x})-\hat{F}(\bar{x}, y)
$$

Now, we add $g(y)$ to both sides of the inequality and take supremum over $\bar{x} \in \mathbb{R}^{d_{x}}$. This gives us

$$
\langle z, x\rangle-r(x)-\hat{F}(x, y)+g(y) \geq \sup _{\bar{x} \in \mathbb{R}^{d_{x}}}[\langle z, \bar{x}\rangle-r(\bar{x})-\hat{F}(\bar{x}, y)+g(y)]=G(z, y)
$$

which together with (36) implies $G(z, y)=\langle z, x\rangle-r(x)-\hat{F}(x, y)+g(y)$.
Next, we use $w+\nabla_{y} \hat{F}(x, y) \in \partial g(y)$, which for arbitrary $\bar{y} \in \mathbb{R}^{d_{y}}$ implies

$$
\begin{aligned}
g(\bar{y}) & \geq g(y)+\left\langle w+\nabla_{y} \hat{F}(x, y), \bar{y}-y\right\rangle \\
& \geq g(y)+\langle w, \bar{y}-y\rangle+\hat{F}(x, \bar{y})-\hat{F}(x, y)
\end{aligned}
$$

where we used the concavity of $\hat{F}(x, y)$ in $y$ in the last inequality. After rearranging we get

$$
g(\bar{y})-\hat{F}(x, \bar{y}) \geq g(y)-\hat{F}(x, y)+\langle w, \bar{y}-y\rangle
$$

Now, we choose arbitrary $\bar{z} \in \mathbb{R}^{d_{x}}$ and add $\langle x, \bar{z}\rangle-r(x)$ to both sides of the inequality, which implies

$$
\begin{aligned}
g(\bar{y})-\hat{F}(x, \bar{y})+\langle x, \bar{z}\rangle-r(x) & \geq g(y)-\hat{F}(x, y)+\langle w, \bar{y}-y\rangle+\langle x, \bar{z}\rangle-r(x) \\
& =G(z, y)+\langle x, \bar{z}-z\rangle+\langle w, \bar{y}-y\rangle
\end{aligned}
$$

where we used $G(z, y)=\langle z, x\rangle-r(x)-\hat{F}(x, y)+g(y)$. Further, using (36) we get

$$
G(z, y)+\langle x, \bar{z}-z\rangle+\langle w, \bar{y}-y\rangle \leq G(\bar{z}, \bar{y})
$$

which holds for arbitrary $\bar{z} \in \mathbb{R}^{d_{x}}, \bar{y} \in \mathbb{R}^{d_{y}}$. Hence, $(x, w) \in \partial G(z, y)$ by the definition of the subdifferential of a convex function.

## B Proof of Lemma 2

From the optimality conditions (5) and the definition of $\hat{F}(x, y)$, it follows that

$$
\begin{aligned}
& -\mu_{x} x^{*}-\nabla_{x} \hat{F}\left(x^{*}, y^{*}\right) \in \partial r\left(x^{*}\right) \\
& -\mu_{y} y^{*}+\nabla_{y} \hat{F}\left(x^{*}, y^{*}\right) \in \partial g\left(y^{*}\right)
\end{aligned}
$$

Using (11) we get

$$
\begin{aligned}
z^{*}-\nabla_{x} \hat{F}\left(x^{*}, y^{*}\right) & \in \partial r\left(x^{*}\right) \\
-\mu_{y} y^{*}+\nabla_{y} \hat{F}\left(x^{*}, y^{*}\right) & \in \partial g\left(y^{*}\right)
\end{aligned}
$$

Using Lemma 1 we get

$$
\left(x^{*},-\mu_{y} y^{*}\right) \in \partial G\left(z^{*}, y^{*}\right)
$$

which together with (11) implies

$$
\left(-\mu_{x}^{-1} z^{*},-\mu_{y} y^{*}\right) \in \partial G\left(z^{*}, y^{*}\right)
$$

The latter condition implies $0 \in \partial P\left(z^{*}, y^{*}\right)$. Hence, $\left(z^{*}, y^{*}\right)$ is indeed a solution of problem (10). The uniqueness of this solution is implied by the strong convexity of the function $P(z, y)$.

## C Proof of Theorem 1

We start with proving two technical lemmas.

## Lemma 5. Under conditions of Theorem 1 the following inequality holds:

$$
\begin{align*}
\frac{1}{\eta_{z}}\left\|z^{k+1}-z^{*}\right\|^{2} & \leq\left(\frac{1}{\eta_{z}}-\mu_{x}^{-1}\right)\left\|z^{k}-z^{*}\right\|^{2}+\mu_{x}^{-1}\left\|z_{f}^{k+1}-z^{*}\right\|^{2} \\
& +\frac{2}{\alpha}\left\langle x_{f}^{k+1}+\mu_{x}^{-1} z_{f}^{k+1},(1-\alpha) z_{f}^{k}+\alpha z^{*}-z_{f}^{k+1}\right\rangle  \tag{37}\\
& +\frac{1}{\alpha}\left(\frac{8}{\mu_{x}}\left\|z_{f}^{k+1}+\frac{\mu_{x}}{2}\left(x_{f}^{k+1}-\mu_{x}^{-1} z_{g}^{k}\right)\right\|^{2}-\frac{\mu_{x}}{8}\left\|x_{f}^{k+1}+\mu_{x}^{-1} z_{g}^{k}\right\|^{2}\right) .
\end{align*}
$$

Proof. Using line 6 of Algorithm 2 we get

$$
\begin{aligned}
\frac{1}{\eta_{z}}\left\|z^{k+1}-z^{*}\right\|^{2} & =\frac{1}{\eta_{z}}\left\|z^{k}-z^{*}\right\|^{2}+\frac{2}{\eta_{z}}\left\langle z^{k+1}-z^{k}, z^{k}-z^{*}\right\rangle+\frac{1}{\eta_{z}}\left\|z^{k+1}-z^{k}\right\|^{2} \\
& =\frac{1}{\eta_{z}}\left\|z^{k}-z^{*}\right\|^{2}+\eta_{z}\left\|\mu_{x}^{-1}\left(z_{f}^{k+1}-z^{k}\right)-\left(x_{f}^{k+1}+\mu_{x}^{-1} z_{f}^{k+1}\right)\right\|^{2} \\
& +2 \mu_{x}^{-1}\left\langle z_{f}^{k+1}-z^{k}, z^{k}-z^{*}\right\rangle-2\left\langle x_{f}^{k+1}+\mu_{x}^{-1} z_{f}^{k+1}, z^{k}-z^{*}\right\rangle
\end{aligned}
$$

Using the parallelogram rule we get

$$
\begin{aligned}
\frac{1}{\eta_{z}}\left\|z^{k+1}-z^{*}\right\|^{2} & =\frac{1}{\eta_{z}}\left\|z^{k}-z^{*}\right\|^{2}+\eta_{z}\left\|\mu_{x}^{-1}\left(z_{f}^{k+1}-z^{k}\right)-\left(x_{f}^{k+1}+\mu_{x}^{-1} z_{f}^{k+1}\right)\right\|^{2} \\
& +\mu_{x}^{-1}\left\|z_{f}^{k+1}-z^{*}\right\|^{2}-\mu_{x}^{-1}\left\|z^{k}-z^{*}\right\|^{2}-\mu_{x}^{-1}\left\|z_{f}^{k+1}-z^{k}\right\|^{2} \\
& -2\left\langle x_{f}^{k+1}+\mu_{x}^{-1} z_{f}^{k+1}, z^{k}-z^{*}\right\rangle
\end{aligned}
$$

Using the inequality $\|a+b\|^{2} \leq 2\|a\|^{2}+2\|b\|^{2}$ we get

$$
\begin{aligned}
\frac{1}{\eta_{z}}\left\|z^{k+1}-z^{*}\right\|^{2} & \leq \frac{1}{\eta_{z}}\left\|z^{k}-z^{*}\right\|^{2}+2 \eta_{z}\left\|x_{f}^{k+1}+\mu_{x}^{-1} z_{f}^{k+1}\right\|^{2}+2 \eta_{z} \mu_{x}^{-2}\left\|z_{f}^{k+1}-z^{k}\right\|^{2} \\
& +\mu_{x}^{-1}\left\|z_{f}^{k+1}-z^{*}\right\|^{2}-\mu_{x}^{-1}\left\|z^{k}-z^{*}\right\|^{2}-\mu_{x}^{-1}\left\|z_{f}^{k+1}-z^{k}\right\|^{2} \\
& -2\left\langle x_{f}^{k+1}+\mu_{x}^{-1} z_{f}^{k+1}, z^{k}-z^{*}\right\rangle
\end{aligned}
$$

Using the definition of $\eta_{z}$ we get

$$
\begin{aligned}
\frac{1}{\eta_{z}}\left\|z^{k+1}-z^{*}\right\|^{2} & \leq \frac{1}{\eta_{z}}\left\|z^{k}-z^{*}\right\|^{2}+\mu_{x}\left\|x_{f}^{k+1}+\mu_{x}^{-1} z_{f}^{k+1}\right\|^{2}+\mu_{x}^{-1}\left\|z_{f}^{k+1}-z^{k}\right\|^{2} \\
& +\mu_{x}^{-1}\left\|z_{f}^{k+1}-z^{*}\right\|^{2}-\mu_{x}^{-1}\left\|z^{k}-z^{*}\right\|^{2}-\mu_{x}^{-1}\left\|z_{f}^{k+1}-z^{k}\right\|^{2} \\
& -2\left\langle x_{f}^{k+1}+\mu_{x}^{-1} z_{f}^{k+1}, z^{k}-z^{*}\right\rangle \\
& =\left(\frac{1}{\eta_{z}}-\mu_{x}^{-1}\right)\left\|z^{k}-z^{*}\right\|^{2}+\mu_{x}\left\|x_{f}^{k+1}+\mu_{x}^{-1} z_{f}^{k+1}\right\|^{2} \\
& +\mu_{x}^{-1}\left\|z_{f}^{k+1}-z^{*}\right\|^{2}-2\left\langle x_{f}^{k+1}+\mu_{x}^{-1} z_{f}^{k+1}, z^{k}-z^{*}\right\rangle
\end{aligned}
$$

From line 4 of Algorithm 2 we get $z^{k}=\alpha^{-1} z_{g}^{k}-(1-\alpha) \alpha^{-1} z_{f}^{k}$ which implies

$$
\begin{aligned}
\frac{1}{\eta_{z}}\left\|z^{k+1}-z^{*}\right\|^{2} & \leq\left(\frac{1}{\eta_{z}}-\mu_{x}^{-1}\right)\left\|z^{k}-z^{*}\right\|^{2}+\mu_{x}\left\|x_{f}^{k+1}+\mu_{x}^{-1} z_{f}^{k+1}\right\|^{2} \\
& +\mu_{x}^{-1}\left\|z_{f}^{k+1}-z^{*}\right\|^{2}-2\left\langle x_{f}^{k+1}+\mu_{x}^{-1} z_{f}^{k+1}, \alpha^{-1} z_{g}^{k}-(1-\alpha) \alpha^{-1} z_{f}^{k}-z^{*}\right\rangle \\
& =\left(\frac{1}{\eta_{z}}-\mu_{x}^{-1}\right)\left\|z^{k}-z^{*}\right\|^{2}+\mu_{x}\left\|x_{f}^{k+1}+\mu_{x}^{-1} z_{f}^{k+1}\right\|^{2}+\mu_{x}^{-1}\left\|z_{f}^{k+1}-z^{*}\right\|^{2} \\
& +\frac{2}{\alpha}\left\langle x_{f}^{k+1}+\mu_{x}^{-1} z_{f}^{k+1},(1-\alpha) z_{f}^{k}+\alpha z^{*}-z_{f}^{k+1}\right\rangle \\
& +\frac{2}{\alpha}\left\langle x_{f}^{k+1}+\mu_{x}^{-1} z_{f}^{k+1}, z_{f}^{k+1}-z_{g}^{k}\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
& =\left(\frac{1}{\eta_{z}}-\mu_{x}^{-1}\right)\left\|z^{k}-z^{*}\right\|^{2}+\mu_{x}\left\|x_{f}^{k+1}+\mu_{x}^{-1} z_{f}^{k+1}\right\|^{2}+\mu_{x}^{-1}\left\|z_{f}^{k+1}-z^{*}\right\|^{2} \\
& +\frac{2}{\alpha}\left\langle x_{f}^{k+1}+\mu_{x}^{-1} z_{f}^{k+1},(1-\alpha) z_{f}^{k}+\alpha z^{*}-z_{f}^{k+1}\right\rangle \\
& +\frac{2 \mu_{x}}{\alpha}\left\langle x_{f}^{k+1}+\mu_{x}^{-1} z_{f}^{k+1}, \mu_{x}^{-1}\left(z_{f}^{k+1}-z_{g}^{k}\right)\right\rangle
\end{aligned}
$$

Using the parallelogram rule we get

$$
\begin{aligned}
\frac{1}{\eta_{z}}\left\|z^{k+1}-z^{*}\right\|^{2} & \leq\left(\frac{1}{\eta_{z}}-\mu_{x}^{-1}\right)\left\|z^{k}-z^{*}\right\|^{2}+\mu_{x}\left\|x_{f}^{k+1}+\mu_{x}^{-1} z_{f}^{k+1}\right\|^{2}+\mu_{x}^{-1}\left\|z_{f}^{k+1}-z^{*}\right\|^{2} \\
& +\frac{2}{\alpha}\left\langle x_{f}^{k+1}+\mu_{x}^{-1} z_{f}^{k+1},(1-\alpha) z_{f}^{k}+\alpha z^{*}-z_{f}^{k+1}\right\rangle \\
& +\frac{\mu_{x}}{\alpha}\left\|x_{f}^{k+1}+2 \mu_{x}^{-1} z_{f}^{k+1}-\mu_{x}^{-1} z_{g}^{k}\right\|^{2}-\frac{\mu_{x}^{-1}}{\alpha}\left\|z_{f}^{k+1}-z_{g}^{k}\right\|^{2}-\frac{\mu_{x}}{\alpha}\left\|x_{f}^{k+1}+\mu_{x}^{-1} z_{g}^{k}\right\|^{2} \\
& =\left(\frac{1}{\eta_{z}}-\mu_{x}^{-1}\right)\left\|z^{k}-z^{*}\right\|^{2}+\mu_{x}\left(1-\alpha^{-1}\right)\left\|x_{f}^{k+1}+\mu_{x}^{-1} z_{f}^{k+1}\right\|^{2}+\mu_{x}^{-1}\left\|z_{f}^{k+1}-z^{*}\right\|^{2} \\
& +\frac{2}{\alpha}\left\langle x_{f}^{k+1}+\mu_{x}^{-1} z_{f}^{k+1},(1-\alpha) z_{f}^{k}+\alpha z^{*}-z_{f}^{k+1}\right\rangle \\
& +\frac{4 \mu_{x}^{-1}}{\alpha}\left\|z_{f}^{k+1}+\frac{\mu_{x}}{2}\left(x_{f}^{k+1}-\mu_{x}^{-1} z_{g}^{k}\right)\right\|^{2}-\frac{\mu_{x}^{-1}}{\alpha}\left\|z_{f}^{k+1}-z_{g}^{k}\right\|^{2}
\end{aligned}
$$

Using the fact that $\alpha^{-1} \geq 1$ we get

$$
\begin{aligned}
\frac{1}{\eta_{z}}\left\|z^{k+1}-z^{*}\right\|^{2} & \leq\left(\frac{1}{\eta_{z}}-\mu_{x}^{-1}\right)\left\|z^{k}-z^{*}\right\|^{2}+\mu_{x}^{-1}\left\|z_{f}^{k+1}-z^{*}\right\|^{2} \\
& +\frac{2}{\alpha}\left\langle x_{f}^{k+1}+\mu_{x}^{-1} z_{f}^{k+1},(1-\alpha) z_{f}^{k}+\alpha z^{*}-z_{f}^{k+1}\right\rangle \\
& +\frac{4 \mu_{x}^{-1}}{\alpha}\left\|z_{f}^{k+1}+\frac{\mu_{x}}{2}\left(x_{f}^{k+1}-\mu_{x}^{-1} z_{g}^{k}\right)\right\|^{2}-\frac{\mu_{x}^{-1}}{\alpha}\left\|z_{f}^{k+1}-z_{g}^{k}\right\|^{2} \\
& =\left(\frac{1}{\eta_{z}}-\mu_{x}^{-1}\right)\left\|z^{k}-z^{*}\right\|^{2}+\mu_{x}^{-1}\left\|z_{f}^{k+1}-z^{*}\right\|^{2} \\
& +\frac{2}{\alpha}\left\langle x_{f}^{k+1}+\mu_{x}^{-1} z_{f}^{k+1},(1-\alpha) z_{f}^{k}+\alpha z^{*}-z_{f}^{k+1}\right\rangle \\
& +\frac{4 \mu_{x}^{-1}}{\alpha}\left\|z_{f}^{k+1}+\frac{\mu_{x}}{2}\left(x_{f}^{k+1}-\mu_{x}^{-1} z_{g}^{k}\right)\right\|^{2} \\
& -\frac{\mu_{x}^{-1}}{\alpha}\left\|z_{f}^{k+1}+\frac{\mu_{x}}{2}\left(x_{f}^{k+1}-\mu_{x}^{-1} z_{g}^{k}\right)-\frac{\mu_{x}}{2}\left(x_{f}^{k+1}-\mu_{x}^{-1} z_{g}^{k}\right)-z_{g}^{k}\right\|^{2}
\end{aligned}
$$

Using the inequality $-\|a+b\|^{2} \leq\|b\|^{2}-\frac{1}{2}\|a\|^{2}$ we get

$$
\begin{aligned}
\frac{1}{\eta_{z}}\left\|z^{k+1}-z^{*}\right\|^{2} & \leq\left(\frac{1}{\eta_{z}}-\mu_{x}^{-1}\right)\left\|z^{k}-z^{*}\right\|^{2}+\mu_{x}^{-1}\left\|z_{f}^{k+1}-z^{*}\right\|^{2} \\
& +\frac{2}{\alpha}\left\langle x_{f}^{k+1}+\mu_{x}^{-1} z_{f}^{k+1},(1-\alpha) z_{f}^{k}+\alpha z^{*}-z_{f}^{k+1}\right\rangle \\
& +\frac{4 \mu_{x}^{-1}}{\alpha}\left\|z_{f}^{k+1}+\frac{\mu_{x}}{2}\left(x_{f}^{k+1}-\mu_{x}^{-1} z_{g}^{k}\right)\right\|^{2} \\
& +\frac{\mu_{x}^{-1}}{\alpha}\left\|z_{f}^{k+1}+\frac{\mu_{x}}{2}\left(x_{f}^{k+1}-\mu_{x}^{-1} z_{g}^{k}\right)\right\|^{2}-\frac{\mu_{x}^{-1}}{2 \alpha}\left\|\frac{\mu_{x}}{2}\left(x_{f}^{k+1}-\mu_{x}^{-1} z_{g}^{k}\right)+z_{g}^{k}\right\|^{2} \\
& =\left(\frac{1}{\eta_{z}}-\mu_{x}^{-1}\right)\left\|z^{k}-z^{*}\right\|^{2}+\mu_{x}^{-1}\left\|z_{f}^{k+1}-z^{*}\right\|^{2} \\
& +\frac{2}{\alpha}\left\langle x_{f}^{k+1}+\mu_{x}^{-1} z_{f}^{k+1},(1-\alpha) z_{f}^{k}+\alpha z^{*}-z_{f}^{k+1}\right\rangle \\
& +\frac{5 \mu_{x}^{-1}}{\alpha}\left\|z_{f}^{k+1}+\frac{\mu_{x}}{2}\left(x_{f}^{k+1}-\mu_{x}^{-1} z_{g}^{k}\right)\right\|^{2}-\frac{\mu_{x}}{8 \alpha}\left\|x_{f}^{k+1}+\mu_{x}^{-1} z_{g}^{k}\right\|^{2}
\end{aligned}
$$

Using the inequality $5 \leq 8$ concludes the proof

Lemma 6. Under conditions of Theorem 1 the following inequality holds:

$$
\begin{align*}
\frac{1}{\eta_{y}}\left\|y^{k+1}-y^{*}\right\|^{2} & \leq\left(\frac{1}{\eta_{y}}-\mu_{y}\right)\left\|y^{k}-y^{*}\right\|^{2}+\mu_{y}\left\|y_{f}^{k+1}-y^{*}\right\|^{2} \\
& +\frac{2}{\alpha}\left\langle w_{f}^{k+1}+\mu_{y} y_{f}^{k+1},(1-\alpha) y_{f}^{k}+\alpha y^{*}-y_{f}^{k+1}\right\rangle  \tag{38}\\
& +\frac{1}{\alpha}\left(\theta_{y}\left\|w_{f}^{k+1}+\mu_{y} y_{f}^{k+1}+\theta_{y}^{-1}\left(y_{f}^{k+1}-y_{g}^{k}\right)\right\|^{2}-\theta_{y}^{-1}\left\|y_{f}^{k+1}-y_{g}^{k}\right\|^{2}\right)
\end{align*}
$$

Proof. Using line 7 of Algorithm 2 we get

$$
\begin{aligned}
\frac{1}{\eta_{y}}\left\|y^{k+1}-y^{*}\right\|^{2} & =\frac{1}{\eta_{y}}\left\|y^{k}-y^{*}\right\|^{2}+\frac{2}{\eta_{y}}\left\langle y^{k+1}-y^{k}, y^{k}-y^{*}\right\rangle+\frac{1}{\eta_{y}}\left\|y^{k+1}-y^{*}\right\|^{2} \\
& =\frac{1}{\eta_{y}}\left\|y^{k}-y^{*}\right\|^{2}+\eta_{y}\left\|\mu_{y}\left(y_{f}^{k+1}-y^{k}\right)-\left(w_{f}^{k+1}+\mu_{y} y_{f}^{k+1}\right)\right\|^{2} \\
& +2 \mu_{y}\left\langle y_{f}^{k+1}-y^{k}, y^{k}-y^{*}\right\rangle-2\left\langle w_{f}^{k+1}+\mu_{y} y_{f}^{k+1}, y^{k}-y^{*}\right\rangle .
\end{aligned}
$$

Using the parallelogram rule we get

$$
\begin{aligned}
& \frac{1}{\eta_{y}}\left\|y^{k+1}-y^{*}\right\|^{2}=\frac{1}{\eta_{y}}\left\|y^{k}-y^{*}\right\|^{2}+\eta_{y}\left\|\mu_{y}\left(y_{f}^{k+1}-y^{k}\right)-\left(w_{f}^{k+1}+\mu_{y} y_{f}^{k+1}\right)\right\|^{2} \\
& \mu_{y}\left\|y_{f}^{k+1}-y^{*}\right\|^{2}-\mu_{y}\left\|y^{k}-y^{*}\right\|^{2}-\mu_{y}\left\|y_{f}^{k+1}-y^{k}\right\|^{2} \\
&-2\left\langle w_{f}^{k+1}+\mu_{y} y_{f}^{k+1}, y^{k}-y^{*}\right\rangle
\end{aligned}
$$

Using the inequality $\|a+b\|^{2} \leq 2\|a\|^{2}+2\|b\|^{2}$ we get

$$
\begin{aligned}
\frac{1}{\eta_{y}}\left\|y^{k+1}-y^{*}\right\|^{2} & =\frac{1}{\eta_{y}}\left\|y^{k}-y^{*}\right\|^{2}+2 \eta_{y}\left\|w_{f}^{k+1}+\mu_{y} y_{f}^{k+1}\right\|^{2}+2 \eta_{y} \mu_{y}^{2}\left\|y_{f}^{k+1}-y^{k}\right\|^{2} \\
& +\mu_{y}\left\|y_{f}^{k+1}-y^{*}\right\|^{2}-\mu_{y}\left\|y^{k}-y^{*}\right\|^{2}-\mu_{y}\left\|y_{f}^{k+1}-y^{k}\right\|^{2} \\
& -2\left\langle w_{f}^{k+1}+\mu_{y} y_{f}^{k+1}, y^{k}-y^{*}\right\rangle
\end{aligned}
$$

Using the definition of $\eta_{y}$ we get

$$
\begin{aligned}
\frac{1}{\eta_{y}}\left\|y^{k+1}-y^{*}\right\|^{2} & \leq \frac{1}{\eta_{y}}\left\|y^{k}-y^{*}\right\|^{2}+2 \eta_{y}\left\|w_{f}^{k+1}+\mu_{y} y_{f}^{k+1}\right\|^{2}+\mu_{y}\left\|y_{f}^{k+1}-y^{k}\right\|^{2} \\
& +\mu_{y}\left\|y_{f}^{k+1}-y^{*}\right\|^{2}-\mu_{y}\left\|y^{k}-y^{*}\right\|^{2}-\mu_{y}\left\|y_{f}^{k+1}-y^{k}\right\|^{2} \\
& -2\left\langle w_{f}^{k+1}+\mu_{y} y_{f}^{k+1}, y^{k}-y^{*}\right\rangle \\
& =\left(\frac{1}{\eta_{y}}-\mu_{y}\right)\left\|y^{k}-y^{*}\right\|^{2}+2 \eta_{y}\left\|w_{f}^{k+1}+\mu_{y} y_{f}^{k+1}\right\|^{2} \\
& +\mu_{y}\left\|y_{f}^{k+1}-y^{*}\right\|^{2}-2\left\langle w_{f}^{k+1}+\mu_{y} y_{f}^{k+1}, y^{k}-y^{*}\right\rangle
\end{aligned}
$$

From line 4 of Algorithm 2 we get $y^{k}=\alpha^{-1} y_{g}^{k}-(1-\alpha) \alpha^{-1} y_{f}^{k}$, which implies

$$
\begin{aligned}
\frac{1}{\eta_{y}}\left\|y^{k+1}-y^{*}\right\|^{2} & \leq\left(\frac{1}{\eta_{y}}-\mu_{y}\right)\left\|y^{k}-y^{*}\right\|^{2}+2 \eta_{y}\left\|w_{f}^{k+1}+\mu_{y} y_{f}^{k+1}\right\|^{2} \\
& +\mu_{y}\left\|y_{f}^{k+1}-y^{*}\right\|^{2}-2\left\langle w_{f}^{k+1}+\mu_{y} y_{f}^{k+1}, \alpha^{-1} y_{g}^{k}-(1-\alpha) \alpha^{-1} y_{f}^{k}-y^{*}\right\rangle \\
& =\left(\frac{1}{\eta_{y}}-\mu_{y}\right)\left\|y^{k}-y^{*}\right\|^{2}+2 \eta_{y}\left\|w_{f}^{k+1}+\mu_{y} y_{f}^{k+1}\right\|^{2}+\mu_{y}\left\|y_{f}^{k+1}-y^{*}\right\|^{2} \\
& +\frac{2}{\alpha}\left\langle w_{f}^{k+1}+\mu_{y} y_{f}^{k+1},(1-\alpha) y_{f}^{k}+\alpha y^{*}-y_{f}^{k+1}\right\rangle \\
& +\frac{2}{\alpha}\left\langle w_{f}^{k+1}+\mu_{y} y_{f}^{k+1}, y_{f}^{k+1}-y_{g}^{k}\right\rangle \\
& =\left(\frac{1}{\eta_{y}}-\mu_{y}\right)\left\|y^{k}-y^{*}\right\|^{2}+2 \eta_{y}\left\|w_{f}^{k+1}+\mu_{y} y_{f}^{k+1}\right\|^{2}+\mu_{y}\left\|y_{f}^{k+1}-y^{*}\right\|^{2}
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{2}{\alpha}\left\langle w_{f}^{k+1}+\mu_{y} y_{f}^{k+1},(1-\alpha) y_{f}^{k}+\alpha y^{*}-y_{f}^{k+1}\right\rangle \\
& +\frac{2 \theta_{y}}{\alpha}\left\langle w_{f}^{k+1}+\mu_{y} y_{f}^{k+1}, \theta_{y}^{-1}\left(y_{f}^{k+1}-y_{g}^{k}\right)\right\rangle
\end{aligned}
$$

Using the parallelogram rule we get

$$
\begin{aligned}
\frac{1}{\eta_{y}}\left\|y^{k+1}-y^{*}\right\|^{2} & \leq\left(\frac{1}{\eta_{y}}-\mu_{y}\right)\left\|y^{k}-y^{*}\right\|^{2}+2 \eta_{y}\left\|w_{f}^{k+1}+\mu_{y} y_{f}^{k+1}\right\|^{2}+\mu_{y}\left\|y_{f}^{k+1}-y^{*}\right\|^{2} \\
& +\frac{2}{\alpha}\left\langle w_{f}^{k+1}+\mu_{y} y_{f}^{k+1},(1-\alpha) y_{f}^{k}+\alpha y^{*}-y_{f}^{k+1}\right\rangle \\
& +\frac{\theta_{y}}{\alpha}\left\|w_{f}^{k+1}+\mu_{y} y_{f}^{k+1}+\theta_{y}^{-1}\left(y_{f}^{k+1}-y_{g}^{k}\right)\right\|^{2} \\
& -\frac{\theta_{y}}{\alpha}\left\|w_{f}^{k+1}+\mu_{y} y_{f}^{k+1}\right\|^{2}-\frac{\theta_{y}^{-1}}{\alpha}\left\|y_{f}^{k+1}-y_{g}^{k}\right\|^{2} \\
& =\left(\frac{1}{\eta_{y}}-\mu_{y}\right)\left\|y^{k}-y^{*}\right\|^{2}+\left(2 \eta_{y}-\alpha^{-1} \theta_{y}\right)\left\|w_{f}^{k+1}+\mu_{y} y_{f}^{k+1}\right\|^{2}+\mu_{y}\left\|y_{f}^{k+1}-y^{*}\right\|^{2} \\
& +\frac{2}{\alpha}\left\langle w_{f}^{k+1}+\mu_{y} y_{f}^{k+1},(1-\alpha) y_{f}^{k}+\alpha y^{*}-y_{f}^{k+1}\right\rangle \\
& +\frac{\theta_{y}}{\alpha}\left\|w_{f}^{k+1}+\mu_{y} y_{f}^{k+1}+\theta_{y}^{-1}\left(y_{f}^{k+1}-y_{g}^{k}\right)\right\|^{2}-\frac{\theta_{y}^{-1}}{\alpha}\left\|y_{f}^{k+1}-y_{g}^{k}\right\|^{2}
\end{aligned}
$$

From the definition of $\eta_{y}$ it follows that $2 \eta_{y} \leq \alpha^{-1} \theta_{y}$. Hence,

$$
\begin{aligned}
\frac{1}{\eta_{y}}\left\|y^{k+1}-y^{*}\right\|^{2} & \leq\left(\frac{1}{\eta_{y}}-\mu_{y}\right)\left\|y^{k}-y^{*}\right\|^{2}+\mu_{y}\left\|y_{f}^{k+1}-y^{*}\right\|^{2} \\
& +\frac{2}{\alpha}\left\langle w_{f}^{k+1}+\mu_{y} y_{f}^{k+1},(1-\alpha) y_{f}^{k}+\alpha y^{*}-y_{f}^{k+1}\right\rangle \\
& +\frac{\theta_{y}}{\alpha}\left\|w_{f}^{k+1}+\mu_{y} y_{f}^{k+1}+\theta_{y}^{-1}\left(y_{f}^{k+1}-y_{g}^{k}\right)\right\|^{2}-\frac{\theta_{y}^{-1}}{\alpha}\left\|y_{f}^{k+1}-y_{g}^{k}\right\|^{2} .
\end{aligned}
$$

Lemma 7. Under conditions of Theorem 1 let $\mathcal{L}^{k}$ be the following Lyapunov function

$$
\begin{equation*}
\mathcal{L}^{k}=\frac{1}{\eta_{z}}\left\|z^{k}-z^{*}\right\|^{2}+\frac{1}{\eta_{y}}\left\|y^{k}-y^{*}\right\|^{2}+\frac{2}{\alpha}\left(P\left(z_{f}^{k}, y_{f}^{k}\right)-P\left(z^{*}, y^{*}\right)\right) \tag{39}
\end{equation*}
$$

Then, the following inequality holds

$$
\begin{equation*}
\mathcal{L}^{k+1} \leq\left(1-\max \left\{\frac{2}{\alpha}, \frac{2 \alpha}{\theta_{y} \mu_{y}}\right\}^{-1}\right) \mathcal{L}^{k} \tag{40}
\end{equation*}
$$

Proof. We start with combining Lemmas 5 and 6 and get

$$
\begin{aligned}
(\mathrm{DISTANCE}) & \leq\left(\frac{1}{\eta_{z}}-\mu_{x}^{-1}\right)\left\|z^{k}-z^{*}\right\|^{2}+\mu_{x}^{-1}\left\|z_{f}^{k+1}-z^{*}\right\|^{2} \\
& +\frac{2}{\alpha}\left\langle x_{f}^{k+1}+\mu_{x}^{-1} z_{f}^{k+1},(1-\alpha) z_{f}^{k}+\alpha z^{*}-z_{f}^{k+1}\right\rangle \\
& +\frac{1}{\alpha}\left(\frac{8}{\mu_{x}}\left\|z_{f}^{k+1}+\frac{\mu_{x}}{2}\left(x_{f}^{k+1}-\mu_{x}^{-1} z_{g}^{k}\right)\right\|^{2}-\frac{\mu_{x}}{8}\left\|x_{f}^{k+1}+\mu_{x}^{-1} z_{g}^{k}\right\|^{2}\right) \\
& +\left(\frac{1}{\eta_{y}}-\mu_{y}\right)\left\|y^{k}-y^{*}\right\|^{2}+\mu_{y}\left\|y_{f}^{k+1}-y^{*}\right\|^{2} \\
& +\frac{2}{\alpha}\left\langle w_{f}^{k+1}+\mu_{y} y_{f}^{k+1},(1-\alpha) y_{f}^{k}+\alpha y^{*}-y_{f}^{k+1}\right\rangle \\
& +\frac{1}{\alpha}\left(\theta_{y}\left\|w_{f}^{k+1}+\mu_{y} y_{f}^{k+1}+\theta_{y}^{-1}\left(y_{f}^{k+1}-y_{g}^{k}\right)\right\|^{2}-\theta_{y}^{-1}\left\|y_{f}^{k+1}-y_{g}^{k}\right\|^{2}\right)
\end{aligned}
$$

where (DISTANCE) is defined as

$$
(\text { DISTANCE })=\frac{1}{\eta_{z}}\left\|z^{k+1}-z^{*}\right\|^{2}+\frac{1}{\eta_{y}}\left\|y^{k+1}-y^{*}\right\|^{2}
$$

Using condition (18) we get

$$
\begin{aligned}
\text { (DISTANCE) } & \leq\left(\frac{1}{\eta_{z}}-\mu_{x}^{-1}\right)\left\|z^{k}-z^{*}\right\|^{2}+\left(\frac{1}{\eta_{y}}-\mu_{y}\right)\left\|y^{k}-y^{*}\right\|^{2} \\
& +\mu_{x}^{-1}\left\|z_{f}^{k+1}-z^{*}\right\|^{2}+\mu_{y}\left\|y_{f}^{k+1}-y^{*}\right\|^{2} \\
& +\frac{2}{\alpha}\left\langle x_{f}^{k+1}+\mu_{x}^{-1} z_{f}^{k+1},(1-\alpha) z_{f}^{k}+\alpha z^{*}-z_{f}^{k+1}\right\rangle \\
& +\frac{2}{\alpha}\left\langle w_{f}^{k+1}+\mu_{y} y_{f}^{k+1},(1-\alpha) y_{f}^{k}+\alpha y^{*}-y_{f}^{k+1}\right\rangle \\
& =\left(\frac{1}{\eta_{z}}-\mu_{x}^{-1}\right)\left\|z^{k}-z^{*}\right\|^{2}+\left(\frac{1}{\eta_{y}}-\mu_{y}\right)\left\|y^{k}-y^{*}\right\|^{2} \\
& +\mu_{x}^{-1}\left\|z_{f}^{k+1}-z^{*}\right\|^{2}+\mu_{y}\left\|y_{f}^{k+1}-y^{*}\right\|^{2} \\
& +\frac{2(1-\alpha)}{\alpha}\left\langle\left[\begin{array}{c}
x_{f}^{k+1} \\
w_{f}^{k+1}
\end{array}\right]+\left[\begin{array}{c}
\mu_{x}^{-1} z_{f}^{k+1} \\
\mu_{y} y_{f}^{k+1}
\end{array}\right],\left[\begin{array}{l}
z_{f}^{k} \\
y_{f}^{k}
\end{array}\right]-\left[\begin{array}{l}
z_{f}^{k+1} \\
y_{f}^{k+1}
\end{array}\right]\right\rangle \\
& +2\left\langle\left[\begin{array}{l}
x_{f}^{k+1} \\
w_{f}^{k+1}
\end{array}\right]+\left[\begin{array}{c}
\mu_{x}^{-1} z_{f}^{k+1} \\
\mu_{y} y_{f}^{k+1}
\end{array}\right],\left[\begin{array}{l}
z^{*} \\
y^{*}
\end{array}\right]-\left[\begin{array}{c}
z_{f}^{k+1} \\
y_{f}^{k+1}
\end{array}\right]\right\rangle .
\end{aligned}
$$

Using condition (18) and Lemma 1 we get

$$
\left[\begin{array}{c}
x_{f}^{k+1} \\
w_{f}^{k+1}
\end{array}\right] \in \partial G\left(z_{f}^{k+1}, y_{f}^{k+1}\right)
$$

Using the definition of function $P(z, y)$ we get

$$
\left[\begin{array}{c}
x_{f}^{k+1}  \tag{41}\\
w_{f}^{k+1}
\end{array}\right]+\left[\begin{array}{c}
\mu_{x}^{-1} z_{f}^{k+1} \\
\mu_{y} y_{f}^{k+1}
\end{array}\right] \in \partial P\left(z_{f}^{k+1}, y_{f}^{k+1}\right)
$$

Hence, using the strong convexity of Function $P(z, y)$ we get

$$
\begin{aligned}
(\text { DISTANCE }) & \leq\left(\frac{1}{\eta_{z}}-\mu_{x}^{-1}\right)\left\|z^{k}-z^{*}\right\|^{2}+\left(\frac{1}{\eta_{y}}-\mu_{y}\right)\left\|y^{k}-y^{*}\right\|^{2} \\
& +\mu_{x}^{-1}\left\|z_{f}^{k+1}-z^{*}\right\|^{2}+\mu_{y}\left\|y_{f}^{k+1}-y^{*}\right\|^{2} \\
& +\frac{2(1-\alpha)}{\alpha}\left(P\left(z_{f}^{k}, y_{f}^{k}\right)-P\left(z_{f}^{k+1}, y_{f}^{k+1}\right)\right) \\
& +2\left(P\left(z^{*}, y^{*}\right)-P\left(z_{f}^{k+1}, y_{f}^{k+1}\right)-\frac{\mu_{x}^{-1}}{2}\left\|z_{f}^{k+1}-z^{*}\right\|^{2}-\frac{\mu_{y}}{2}\left\|y_{f}^{k+1}-y^{*}\right\|^{2}\right) \\
& =\left(\frac{1}{\eta_{z}}-\mu_{x}^{-1}\right)\left\|z^{k}-z^{*}\right\|^{2}+\left(\frac{1}{\eta_{y}}-\mu_{y}\right)\left\|y^{k}-y^{*}\right\|^{2} \\
& +\frac{2(1-\alpha)}{\alpha}\left(P\left(z_{f}^{k}, y_{f}^{k}\right)-P\left(z^{*}, y^{*}\right)\right)-\frac{2}{\alpha}\left(P\left(z_{f}^{k+1}, y_{f}^{k+1}\right)-P\left(z^{*}, y^{*}\right)\right)
\end{aligned}
$$

After rearranging and using the definition of (DISTANCE) and the definition of $\mathcal{L}^{k}$ we get

$$
\begin{aligned}
\mathcal{L}^{k+1} & \leq\left(\frac{1}{\eta_{z}}-\mu_{x}^{-1}\right)\left\|z^{k}-z^{*}\right\|^{2}+\left(\frac{1}{\eta_{y}}-\mu_{y}\right)\left\|y^{k}-y^{*}\right\|^{2} \\
& +\frac{2(1-\alpha)}{\alpha}\left(P\left(z_{f}^{k}, y_{f}^{k}\right)-P\left(z^{*}, y^{*}\right)\right) \\
& \leq\left(1-\max \left\{2, \frac{1}{\alpha}, \frac{2 \alpha}{\theta_{y} \mu_{y}}\right\}^{-1}\right) \mathcal{L}^{k} \\
& \leq\left(1-\max \left\{\frac{2}{\alpha}, \frac{2 \alpha}{\theta_{y} \mu_{y}}\right\}^{-1}\right) \mathcal{L}^{k} .
\end{aligned}
$$

Now, we are ready to prove Theorem 1 .
Proof of Theorem 1. After unrolling the recurrence from Lemma 7 we get

$$
\frac{1}{\eta_{z}}\left\|z^{K}-z^{*}\right\|^{2}+\frac{1}{\eta_{y}}\left\|y^{K}-y^{*}\right\|^{2} \leq\left(1-\max \left\{\frac{2}{\alpha}, \frac{2 \alpha}{\theta_{y} \mu_{y}}\right\}^{-1}\right)^{K} \mathcal{L}^{0}
$$

Using Lemma 2 we get

$$
\begin{aligned}
\left(1-\max \left\{\frac{2}{\alpha}, \frac{2 \alpha}{\theta_{y} \mu_{y}}\right\}^{-1}\right)^{K} \mathcal{L}^{0} & \geq \frac{1}{\eta_{z}}\left\|z^{K}+\mu_{x} z^{*}\right\|^{2}+\frac{1}{\eta_{y}}\left\|y^{K}-y^{*}\right\|^{2} \\
& =\frac{\mu_{x}^{2}}{\eta_{z}}\left\|\mu_{x}^{-1} z^{K}+z^{*}\right\|^{2}+\frac{1}{\eta_{y}}\left\|y^{K}-y^{*}\right\|^{2} \\
& \geq \min \left\{\eta_{z}^{-1} \mu_{x}^{2}, \eta_{y}^{-1}\right\}\left(\left\|\mu_{x}^{-1} z^{K}+z^{*}\right\|^{2}+\left\|y^{K}-y^{*}\right\|^{2}\right)
\end{aligned}
$$

After rearranging we get

$$
\left\|\mu_{x}^{-1} z^{K}+z^{*}\right\|^{2}+\left\|y^{K}-y^{*}\right\|^{2} \leq C\left(1-\max \left\{\frac{2}{\alpha}, \frac{2 \alpha}{\theta_{y} \mu_{y}}\right\}^{-1}\right)^{K}
$$

where $C$ is given as

$$
C=\max \left\{\eta_{z} \mu_{x}^{-2}, \eta_{y}\right\} \mathcal{L}^{0}
$$

Hence, $\left(\left\|\mu_{x}^{-1} z^{K}+z^{*}\right\|^{2}+\left\|y^{K}-y^{*}\right\|^{2}\right) \leq \epsilon$ if the number of iterations $K$ satisfies

$$
K \geq \max \left\{\frac{2}{\alpha}, \frac{2 \alpha}{\theta_{y} \mu_{y}}\right\} \log \frac{C}{\epsilon}
$$

This concludes the proof.

## D Proof of Theorem 2

We start the proof with three technical lemmas.
Lemma 8. Under assumptions of Theorem 2 the following inequality holds:

$$
\begin{equation*}
96 M^{2}\left\|u^{0}-u^{*}\right\|^{2}+12\left\|a^{0}+b^{0}\right\|^{2} \leq 288 M^{2}\left\|u^{-1}-u^{*}\right\|^{2} \tag{42}
\end{equation*}
$$

Proof. From Line 3 of Algorithm 3 it follows that

$$
u^{0}=\mathrm{J}_{\lambda B}\left(u^{-1}-\lambda a^{-1}\right),
$$

where $a^{-1}=A\left(u^{-1}\right)$. Vector $u^{*}$ is the solution to problem (22). Hence, there exists $b^{*} \in B\left(u^{*}\right)$ such that $a^{*}+b^{*}=0$, where $a^{*}=A\left(u^{*}\right)$. This implies

$$
u^{*}=\mathrm{J}_{\lambda B}\left(u^{*}-\lambda a^{*}\right)
$$

Using the firm non-expansiveness of $\mathrm{J}_{\lambda B}$ we get

$$
\begin{aligned}
\left\|u^{0}-u^{*}\right\|^{2} & =\left\|\mathrm{J}_{\lambda B}\left(u^{-1}-\lambda a^{-1}\right)-\mathrm{J}_{\lambda B}\left(u^{*}-\lambda a^{*}\right)\right\|^{2} \\
& \leq\left\|\left(u^{-1}-\lambda a^{-1}\right)-\left(u^{*}-\lambda a^{*}\right)\right\|^{2} \\
& -\left\|\mathrm{J}_{\lambda B}\left(u^{-1}-\lambda a^{-1}\right)-\mathrm{J}_{\lambda B}\left(u^{*}-\lambda a^{*}\right)-\left(u^{-1}-\lambda a^{-1}\right)+\left(u^{*}-\lambda a^{*}\right)\right\|^{2} \\
& =\left\|u^{-1}-u^{*}-\lambda\left(a^{-1}-a^{*}\right)\right\|^{2}-\left\|\left(u^{*}-\lambda a^{*}-u^{*}\right)-\left(u^{-1}-\lambda a^{-1}-u^{0}\right)\right\|^{2} \\
& =\left\|u^{-1}-u^{*}-\lambda\left(a^{-1}-a^{*}\right)\right\|^{2}-\left\|\lambda a^{*}+\left(u^{-1}-\lambda a^{-1}-u^{0}\right)\right\|^{2}
\end{aligned}
$$

Using line 5 of Algorithm 3 we get

$$
\begin{aligned}
\left\|u^{0}-u^{*}\right\|^{2} & \leq\left\|u^{-1}-u^{*}-\lambda\left(a^{-1}-a^{*}\right)\right\|^{2}-\left\|\lambda a^{*}+\lambda b^{0}\right\|^{2} \\
& =\left\|u^{-1}-u^{*}-\lambda\left(a^{-1}-a^{*}\right)\right\|^{2}-\left\|\lambda\left(a^{*}-a^{0}\right)+\lambda\left(a^{0}+b^{0}\right)\right\|^{2}
\end{aligned}
$$

Using the inequality $\left\|\lambda\left(a^{*}-a^{0}\right)+\lambda\left(a^{0}+b^{0}\right)\right\|^{2} \geq \frac{\lambda^{2}}{2}\left\|a^{0}+b^{0}\right\|^{2}-\lambda^{2}\left\|a^{0}-a^{*}\right\|^{2}$ we get

$$
\left\|u^{0}-u^{*}\right\|^{2} \leq\left\|u^{-1}-u^{*}-\lambda\left(a^{-1}-a^{*}\right)\right\|^{2}-\frac{\lambda^{2}}{2}\left\|a^{0}+b^{0}\right\|^{2}+\lambda^{2}\left\|a^{0}-a^{*}\right\|^{2}
$$

Using the inequality $\left\|u^{-1}-u^{*}-\lambda\left(a^{-1}-a^{*}\right)\right\|^{2} \leq 2\left\|u^{-1}-u^{*}\right\|^{2}+2 \lambda^{2}\left\|a^{-1}-a^{*}\right\|^{2}$ we get

$$
\left\|u^{0}-u^{*}\right\|^{2} \leq 2\left\|u^{-1}-u^{*}\right\|^{2}+2 \lambda^{2}\left\|a^{-1}-a^{*}\right\|^{2}-\frac{\lambda^{2}}{2}\left\|a^{0}+b^{0}\right\|^{2}+\lambda^{2}\left\|a^{0}-a^{*}\right\|^{2}
$$

Using the $M$-Lipschitzness of $A(u)$ we get

$$
\left\|u^{0}-u^{*}\right\|^{2} \leq 2\left\|u^{-1}-u^{*}\right\|^{2}+2 \lambda^{2} M^{2}\left\|u^{-1}-u^{*}\right\|^{2}-\frac{\lambda^{2}}{2}\left\|a^{0}+b^{0}\right\|^{2}+\lambda^{2} M^{2}\left\|u^{0}-u^{*}\right\|^{2}
$$

After rearranging we get

$$
\left(1-\lambda^{2} M^{2}\right)\left\|u^{0}-u^{*}\right\|^{2}+\frac{\lambda^{2}}{2}\left\|a^{0}+b^{0}\right\|^{2} \leq 2\left(1+\lambda^{2} M^{2}\right)\left\|u^{-1}-u^{*}\right\|^{2}
$$

Plugging the definition of $\lambda$ gives

$$
\frac{4}{5}\left\|u^{0}-u^{*}\right\|^{2}+\frac{1}{10 M^{2}}\left\|a^{0}+b^{0}\right\|^{2} \leq \frac{12}{5}\left\|u^{-1}-u^{*}\right\|^{2}
$$

Multiplying both sides of the inequality by $120 M^{2}$ gives

$$
96 M^{2}\left\|u^{0}-u^{*}\right\|^{2}+12\left\|a^{0}+b^{0}\right\|^{2} \leq M^{2}\left\|u^{-1}-u^{*}\right\|^{2}
$$

Lemma 9. Under conditions of Theorem 2 the following equality holds:

$$
\begin{equation*}
\prod_{t=0}^{T-1}\left(1-\beta_{t}\right)=\frac{2}{(T+1)(T+2)} \tag{43}
\end{equation*}
$$

Proof.

$$
\prod_{t=0}^{T-1}\left(1-\beta_{t}\right)=\prod_{t=0}^{T-1}\left(1-\frac{2}{t+3}\right)=\prod_{t=0}^{T-1} \frac{t+1}{t+3}=\frac{\prod_{t=1}^{T} t}{\prod_{t=3}^{T+2} t}=\frac{2}{(T+1)(T+2)}
$$

Lemma 10. Under conditions of Theorem 2 let $\mathcal{U}^{t}$ be the following Lyapunov function

$$
\begin{equation*}
\mathcal{U}^{t}=\left\langle a^{t}+b^{t}, u^{t}-u^{0}\right\rangle+\frac{\lambda}{2 \beta_{t}}\left\|a^{t}+b^{t}\right\|^{2} \tag{44}
\end{equation*}
$$

Then, the following inequality holds for all $t \in\{0,1,2, \ldots, T-1\}$ :

$$
\begin{equation*}
\mathcal{U}^{t+1} \leq\left(1-\beta_{t}\right) \mathcal{U}^{t} \tag{45}
\end{equation*}
$$

Proof. We start with the monotonicity property of operators $A(u)$ and $B(u)$ :

$$
\begin{equation*}
\left\langle u^{t+1}-u^{t},\left(a^{t+1}+b^{t+1}\right)-\left(a^{t}+b^{t}\right)\right\rangle \geq 0 \tag{46}
\end{equation*}
$$

where $t \in\{0,1,2, \ldots, T-1\}$. From line 10 of Algorithm 3 we can conclude that

$$
\begin{equation*}
u^{t+1}=u^{t}+\beta_{t}\left(u^{0}-u^{t}\right)-\lambda\left(a^{t+1 / 2}+b^{t+1}\right), \tag{47}
\end{equation*}
$$

where $a^{t+1 / 2}=A\left(u^{t+1 / 2}\right)$. From this we also conclude that

$$
\begin{equation*}
u^{t+1}=u^{t}+\left(1-\beta_{t}\right)^{-1}\left[\beta_{t}\left(u^{0}-u^{t+1}\right)-\lambda\left(a^{t+1 / 2}+b^{t+1}\right)\right] \tag{48}
\end{equation*}
$$

Plugging (47) and 48) into (46) gives

$$
\begin{aligned}
0 & \leq\left(1-\beta_{t}\right)^{-1}\left\langle a^{t+1}+b^{t+1}, \beta_{t}\left(u^{0}-u^{t+1}\right)-\lambda\left(a^{t+1 / 2}+b^{t+1}\right)\right\rangle \\
& -\left\langle a^{t}+b^{t}, \beta_{t}\left(u^{0}-u^{t}\right)-\lambda\left(a^{t+1 / 2}+b^{t+1}\right)\right\rangle \\
& =-\left(1-\beta_{t}\right)^{-1}\left[\beta_{t}\left\langle a^{t+1}+b^{t+1}, u^{t+1}-u^{0}\right\rangle+\lambda\left\langle a^{t+1}+b^{t+1}, a^{t+1 / 2}+b^{t+1}\right\rangle\right] \\
& +\beta_{t}\left\langle a^{t}+b^{t}, u^{t}-u^{0}\right\rangle+\lambda\left\langle a^{t}+b^{t}, a^{t+1 / 2}+b^{t+1}\right\rangle
\end{aligned}
$$

Using the parallelogram rule we get

$$
\begin{aligned}
0 & \leq-\left(1-\beta_{t}\right)^{-1} \beta_{t}\left\langle a^{t+1}+b^{t+1}, u^{t+1}-u^{0}\right\rangle \\
& -\frac{\left(1-\beta_{t}\right)^{-1} \lambda}{2}\left[\left\|a^{t+1}+b^{t+1}\right\|^{2}+\left\|a^{t+1 / 2}+b^{t+1}\right\|^{2}-\left\|a^{t+1}-a^{t+1 / 2}\right\|^{2}\right] \\
& +\beta_{t}\left\langle a^{t}+b^{t}, u^{t}-u^{0}\right\rangle \\
& +\frac{\lambda}{2}\left[\left\|a^{t}+b^{t}\right\|^{2}+\left\|a^{t+1 / 2}+b^{t+1}\right\|^{2}-\left\|a^{t}+b^{t}-\left(a^{t+1 / 2}+b^{t+1}\right)\right\|^{2}\right] \\
& =\beta_{t}\left\langle a^{t}+b^{t}, u^{t}-u^{0}\right\rangle+\frac{\lambda}{2}\left\|a^{t}+b^{t}\right\|^{2} \\
& -\left(1-\beta_{t}\right)^{-1}\left[\beta_{t}\left\langle a^{t+1}+b^{t+1}, u^{t+1}-u^{0}\right\rangle+\frac{\lambda}{2}\left\|a^{t+1}+b^{t+1}\right\|^{2}\right] \\
& +\frac{\lambda\left(1-\left(1-\beta_{t}\right)^{-1}\right)}{2}\left\|a^{t+1 / 2}+b^{t+1}\right\|^{2}+\frac{\lambda\left(1-\beta_{t}\right)^{-1}}{2}\left\|a^{t+1}-a^{t+1 / 2}\right\|^{2} \\
& -\frac{\lambda}{2}\left\|a^{t}+b^{t}-\left(a^{t+1 / 2}+b^{t+1}\right)\right\|^{2} \\
& =\beta_{t}\left\langle a^{t}+b^{t}, u^{t}-u^{0}\right\rangle+\frac{\lambda}{2}\left\|a^{t}+b^{t}\right\|^{2} \\
& -\left(1-\beta_{t}\right)^{-1}\left[\beta_{t}\left\langle a^{t+1}+b^{t+1}, u^{t+1}-u^{0}\right\rangle+\frac{\lambda}{2}\left\|a^{t+1}+b^{t+1}\right\|^{2}\right] \\
& -\frac{\lambda \beta_{t}\left(1-\beta_{t}\right)^{-1}}{2}\left\|a^{t+1 / 2}+b^{t+1}\right\|^{2}+\frac{\lambda\left(1-\beta_{t}\right)^{-1}}{2}\left\|a^{t+1}-a^{t+1 / 2}\right\|^{2} \\
& -\frac{\lambda}{2}\left\|a^{t}+b^{t}-\left(a^{t+1 / 2}+b^{t+1}\right)\right\|^{2} .
\end{aligned}
$$

Now, we divide both sides of the inequality by $\beta_{t}$ and get

$$
\begin{aligned}
0 & \leq\left\langle a^{t}+b^{t}, u^{t}-u^{0}\right\rangle+\frac{\lambda}{2 \beta_{t}}\left\|a^{t}+b^{t}\right\|^{2} \\
& -\left(1-\beta_{t}\right)^{-1}\left[\left\langle a^{t+1}+b^{t+1}, u^{t+1}-u^{0}\right\rangle+\frac{\lambda}{2 \beta_{t}}\left\|a^{t+1}+b^{t+1}\right\|^{2}\right] \\
& -\frac{\lambda}{2\left(1-\beta_{t}\right)}\left\|a^{t+1 / 2}+b^{t+1}\right\|^{2}+\frac{\lambda}{2 \beta_{t}\left(1-\beta_{t}\right)}\left\|a^{t+1}-a^{t+1 / 2}\right\|^{2} \\
& -\frac{\lambda}{2 \beta_{t}}\left\|a^{t}+b^{t}-\left(a^{t+1 / 2}+b^{t+1}\right)\right\|^{2}
\end{aligned}
$$

using the inequality $\left\|a^{t+1 / 2}+b^{t+1}\right\|^{2} \geq \frac{1}{2}\left\|a^{t+1}+b^{t+1}\right\|^{2}-\left\|a^{t+1}-a^{t+1 / 2}\right\|^{2}$ we get

$$
\begin{aligned}
0 & \leq\left\langle a^{t}+b^{t}, u^{t}-u^{0}\right\rangle+\frac{\lambda}{2 \beta_{t}}\left\|a^{t}+b^{t}\right\|^{2} \\
& -\left(1-\beta_{t}\right)^{-1}\left[\left\langle a^{t+1}+b^{t+1}, u^{t+1}-u^{0}\right\rangle+\frac{\lambda}{2 \beta_{t}}\left\|a^{t+1}+b^{t+1}\right\|^{2}\right] \\
& -\frac{\lambda}{4\left(1-\beta_{t}\right)}\left\|a^{t+1}+b^{t+1}\right\|^{2}+\frac{\lambda}{2\left(1-\beta_{t}\right)}\left\|a^{t+1 / 2}+a^{t+1}\right\|^{2}+\frac{\lambda}{2 \beta_{t}\left(1-\beta_{t}\right)}\left\|a^{t+1}-a^{t+1 / 2}\right\|^{2}
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{\lambda}{2 \beta_{t}}\left\|a^{t}+b^{t}-\left(a^{t+1 / 2}+b^{t+1}\right)\right\|^{2} \\
& =\left\langle a^{t}+b^{t}, u^{t}-u^{0}\right\rangle+\frac{\lambda}{2 \beta_{t}}\left\|a^{t}+b^{t}\right\|^{2} \\
& -\left(1-\beta_{t}\right)^{-1}\left[\left\langle a^{t+1}+b^{t+1}, u^{t+1}-u^{0}\right\rangle+\frac{\lambda}{2}\left(\frac{1}{\beta_{t}}+\frac{1}{2}\right)\left\|a^{t+1}+b^{t+1}\right\|^{2}\right] \\
& +\frac{\lambda}{2 \beta_{t}}\left(\frac{1+\beta_{t}}{1-\beta_{t}}\left\|a^{t+1 / 2}+a^{t+1}\right\|^{2}-\left\|a^{t}+b^{t}-\left(a^{t+1 / 2}+b^{t+1}\right)\right\|^{2}\right)
\end{aligned}
$$

From the definition of $\beta_{t}$ it follows that $\frac{1}{\beta_{t+1}}=\frac{1}{\beta_{t}}+\frac{1}{2}$ and $\frac{1+\beta_{t}}{1-\beta_{t}} \leq 5$. Hence,

$$
\begin{aligned}
0 & \leq\left\langle a^{t}+b^{t}, u^{t}-u^{0}\right\rangle+\frac{\lambda}{2 \beta_{t}}\left\|a^{t}+b^{t}\right\|^{2} \\
& -\left(1-\beta_{t}\right)^{-1}\left[\left\langle a^{t+1}+b^{t+1}, u^{t+1}-u^{0}\right\rangle+\frac{\lambda}{2 \beta_{t+1}}\left\|a^{t+1}+b^{t+1}\right\|^{2}\right] \\
& +\frac{\lambda}{2 \beta_{t}}\left(5\left\|a^{t+1 / 2}+a^{t+1}\right\|^{2}-\left\|a^{t}+b^{t}-\left(a^{t+1 / 2}+b^{t+1}\right)\right\|^{2}\right)
\end{aligned}
$$

Using the $M$-Lipschitzness of $A(u)$ we get

$$
\begin{aligned}
0 & \leq\left\langle a^{t}+b^{t}, u^{t}-u^{0}\right\rangle+\frac{\lambda}{2 \beta_{t}}\left\|a^{t}+b^{t}\right\|^{2} \\
& -\left(1-\beta_{t}\right)^{-1}\left[\left\langle a^{t+1}+b^{t+1}, u^{t+1}-u^{0}\right\rangle+\frac{\lambda}{2 \beta_{t+1}}\left\|a^{t+1}+b^{t+1}\right\|^{2}\right] \\
& +\frac{\lambda}{2 \beta_{t}}\left(5 M^{2}\left\|u^{t+1 / 2}+u^{t+1}\right\|^{2}-\left\|a^{t}+b^{t}-\left(a^{t+1 / 2}+b^{t+1}\right)\right\|^{2}\right)
\end{aligned}
$$

From lines 7 and 10 of Algorithm 3 it follows that

$$
u^{t+1 / 2}-u^{t}=\lambda\left(a^{t}+b^{t}-\left(a^{t+1 / 2}+b^{t+1}\right)\right)
$$

Hence,

$$
\begin{aligned}
0 & \leq\left\langle a^{t}+b^{t}, u^{t}-u^{0}\right\rangle+\frac{\lambda}{2 \beta_{t}}\left\|a^{t}+b^{t}\right\|^{2} \\
& -\left(1-\beta_{t}\right)^{-1}\left[\left\langle a^{t+1}+b^{t+1}, u^{t+1}-u^{0}\right\rangle+\frac{\lambda}{2 \beta_{t+1}}\left\|a^{t+1}+b^{t+1}\right\|^{2}\right] \\
& +\frac{\lambda}{2 \beta_{t}}\left(5 M^{2} \lambda^{2}-1\right)\left\|a^{t}+b^{t}-\left(a^{t+1 / 2}+b^{t+1}\right)\right\|^{2}
\end{aligned}
$$

Using the definition of $\lambda$ we get

$$
\begin{aligned}
0 & \leq\left\langle a^{t}+b^{t}, u^{t}-u^{0}\right\rangle+\frac{\lambda}{2 \beta_{t}}\left\|a^{t}+b^{t}\right\|^{2} \\
& -\left(1-\beta_{t}\right)^{-1}\left[\left\langle a^{t+1}+b^{t+1}, u^{t+1}-u^{0}\right\rangle+\frac{\lambda}{2 \beta_{t+1}}\left\|a^{t+1}+b^{t+1}\right\|^{2}\right] .
\end{aligned}
$$

Rearranging and multiplying both sides of the inequality by $\left(1-\beta_{t}\right)$ concludes the proof.

Now, we are ready to prove Theorem 2
Proof of Theorem 2, Unrolling the recurrence from Lemma 10we get

$$
\mathcal{U}^{T} \leq \prod_{t=0}^{T-1} \mathcal{U}^{0}
$$

Using Lemma 9 we get

$$
\mathcal{U}^{T} \leq \frac{2}{(T+1)(T+2)} \mathcal{U}^{0}
$$

Using the definition of $\mathcal{U}^{t}$ we get

$$
\frac{\lambda}{\beta_{0}(T+1)(T+2)}\left\|a^{0}+b^{0}\right\|^{2} \geq\left\langle a^{T}+b^{T}, u^{T}-u^{0}\right\rangle+\frac{\lambda}{2 \beta_{T}}\left\|a^{T}+b^{T}\right\|^{2}
$$

Using the definition of $\beta_{t}$ we get

$$
\frac{3 \lambda}{2(T+1)(T+2)}\left\|a^{0}+b^{0}\right\|^{2} \geq\left\langle a^{T}+b^{T}, u^{T}-u^{0}\right\rangle+\frac{\lambda(T+3)}{4}\left\|a^{T}+b^{T}\right\|^{2}
$$

Vector $u^{*}$ is the solution to problem (22). Hence, there exists $b^{*} \in B\left(u^{*}\right), a^{*}=A\left(u^{*}\right)$ such that $a^{*}+b^{*}=0$. From the monotonicity assumption it follows that

$$
\left\langle a^{T}+b^{T}, u^{T}-u^{*}\right\rangle=\left\langle a^{T}+b^{T}-\left(a^{*}+b^{*}\right), u^{T}-u^{*}\right\rangle \geq 0
$$

Hence,

$$
\frac{3 \lambda}{2(T+1)(T+2)}\left\|a^{0}+b^{0}\right\|^{2} \geq\left\langle a^{T}+b^{T}, u^{*}-u^{0}\right\rangle+\frac{\lambda(T+3)}{4}\left\|a^{T}+b^{T}\right\|^{2}
$$

Using the Young's inequality we get

$$
\begin{aligned}
\frac{3 \lambda}{2(T+1)(T+2)}\left\|a^{0}+b^{0}\right\|^{2} & \geq-\frac{\lambda(T+3)}{8}\left\|a^{T}+b^{T}\right\|^{2}-\frac{2}{\lambda(T+3)}\left\|u^{0}-u^{*}\right\|^{2} \\
& +\frac{\lambda(T+3)}{4}\left\|a^{T}+b^{T}\right\|^{2} \\
& =\frac{\lambda(T+3)}{8}\left\|a^{T}+b^{T}\right\|^{2}-\frac{2}{\lambda(T+3)}\left\|u^{0}-u^{*}\right\|^{2}
\end{aligned}
$$

After rearranging we get

$$
\frac{\lambda(T+3)}{8}\left\|a^{T}+b^{T}\right\|^{2} \leq \frac{2}{\lambda(T+3)}\left\|u^{0}-u^{*}\right\|^{2}+\frac{3 \lambda}{2(T+1)(T+2)}\left\|a^{0}+b^{0}\right\|^{2}
$$

Multiplying both sides of the inequality by $\frac{8}{\lambda(T+3)}$ gives

$$
\left\|a^{T}+b^{T}\right\|^{2} \leq \frac{8}{\lambda^{2}(T+3)^{2}}\left\|u^{0}-u^{*}\right\|^{2}+\frac{12 \lambda}{(T+1)(T+2)(T+3)}\left\|a^{0}+b^{0}\right\|^{2}
$$

Plugging the definition of $\lambda$ gives

$$
\begin{aligned}
\left\|a^{T}+b^{T}\right\|^{2} & \leq \frac{40 M^{2}}{(T+3)^{2}}\left\|u^{0}-u^{*}\right\|^{2}+\frac{12}{(T+1)(T+2)(T+3)}\left\|a^{0}+b^{0}\right\|^{2} \\
& \leq \frac{1}{(T+1)^{2}}\left(96 M^{2}\left\|u^{0}-u^{*}\right\|^{2}+12\left\|a^{0}+b^{0}\right\|^{2}\right)
\end{aligned}
$$

Using Lemma 8 we get

$$
\left\|a^{T}+b^{T}\right\|^{2} \leq \frac{288 M^{2}}{(T+1)^{2}}\left\|u^{-1}-u^{*}\right\|^{2}
$$

## E Proof of Lemma 3

Monotonicity of $A^{k}(u)$ can be verified trivially. Maximal monotonicity of $B(u)$ follows from the fact that it is equal to the subdifferential of a convex function $\left(\gamma_{x}^{-1 / 2} x, \gamma_{y}^{-1 / 2} y\right) \mapsto r(x)+g(y)$ and (Rockafellar and Wets, 2009, Theorem 12.17).
Now, we prove Lipschitzness of operator $A^{k}(u)$. Denoting $u_{1}=\left(\gamma_{x}^{-1 / 2} x_{1}, \gamma_{y}^{-1 / 2} y_{1}\right)$ and $u_{2}=$ $\left(\gamma_{x}^{-1 / 2} x_{2}, \gamma_{y}^{-1 / 2} y_{2}\right)$ gives

$$
\begin{aligned}
\left\|A^{k}\left(u_{1}\right)-A^{k}\left(u_{2}\right)\right\|^{2} & \leq \gamma_{x}\left\|a_{x}^{k}\left(x_{1}, y_{1}\right)-a_{x}^{k}\left(x_{2}, y_{2}\right)\right\|^{2}+\gamma_{y}\left\|a_{y}^{k}\left(x_{1}, y_{1}\right)-a_{y}^{k}\left(x_{2}, y_{2}\right)\right\|^{2} \\
& \leq 2 \gamma_{x}\left\|a_{x}^{k}\left(x_{1}, y_{1}\right)-a_{x}^{k}\left(x_{1}, y_{2}\right)\right\|^{2}+2 \gamma_{x}\left\|a_{x}^{k}\left(x_{1}, y_{2}\right)-a_{x}^{k}\left(x_{2}, y_{2}\right)\right\|^{2} \\
& +2 \gamma_{y}\left\|a_{y}^{k}\left(x_{1}, y_{1}\right)-a_{y}^{k}\left(x_{1}, y_{2}\right)\right\|^{2}+2 \gamma_{y}\left\|a_{y}^{k}\left(x_{1}, y_{2}\right)-a_{y}^{k}\left(x_{2}, y_{2}\right)\right\|^{2}
\end{aligned}
$$

Using the definition of operators $a_{x}^{k}(x, y)$ and $a_{y}^{k}(x, y)$, the definition of function $\hat{F}(x, y)$ and Assumptions 1 to 3 we get

$$
\begin{aligned}
\left\|A^{k}\left(u_{1}\right)-A^{k}\left(u_{2}\right)\right\|^{2} & \leq 2 \gamma_{x} L^{2}\left\|y_{1}-y_{2}\right\|^{2}+2 \gamma_{x} L^{2}\left\|x_{1}-x_{2}\right\|^{2} \\
& +2 \gamma_{y}\left(L+\theta_{y}^{-1}\right)^{2}\left\|y_{1}-y_{2}\right\|^{2}+2 \gamma_{y} L^{2}\left\|x_{1}-x_{2}\right\|^{2} \\
& =2\left(\gamma_{x} \gamma_{y} L^{2}+\gamma_{y}^{2}\left(L+\theta_{y}^{-1}\right)^{2}\right) \gamma_{y}^{-1}\left\|y_{1}-y_{2}\right\|^{2} \\
& +2\left(\gamma_{x}^{2} L^{2}+\gamma_{x} \gamma_{y} L^{2}\right) \gamma_{x}^{-1}\left\|x_{1}-x_{2}\right\|^{2} \\
& \leq 4 \max \left\{\gamma_{x}^{2} L^{2}, \gamma_{x} \gamma_{y} L^{2}, \gamma_{y}^{2}\left(L+\theta_{y}^{-1}\right)^{2}\right\}\left\|u_{1}-u_{2}\right\|^{2} \\
& =4 \max \left\{\gamma_{x}^{2} L^{2}, \gamma_{y}^{2}\left(L+\theta_{y}^{-1}\right)^{2}\right\}\left\|u_{1}-u_{2}\right\|^{2} .
\end{aligned}
$$

## F Proof of Lemma 4

To prove this lemma it is sufficient to show that condition is satisfied when $t=T$, where $T$ is given by (33).
Let $\left(x^{k, *}, y^{k, *}\right)$ be the solution of the monotone inclusion problem

$$
0 \in A^{k}\left(\left(\gamma_{x}^{-1 / 2} x^{k, *}, \gamma_{y}^{-1 / 2} y^{k, *}\right)\right)+B\left(\left(\gamma_{x}^{-1 / 2} x^{k, *} \gamma_{y}^{-1 / 2} y^{k, *}\right)\right)
$$

Note, that this solution always exists. One can easily show that operator $A^{k}(u)$ is strongly monotone, which implies the following inequality:

$$
\begin{aligned}
\left\langle x^{k, t}-x^{k, *}, a_{x}^{k}\left(x^{k, t}, y^{k, t}\right)+b_{x}^{k, t}\right\rangle & +\left\langle y^{k, t}-y^{k, *}, a_{y}^{k}\left(x^{k, t}, y^{k, t}\right)+b_{y}^{k, t}\right\rangle \geq \\
& \geq \frac{\mu_{x}}{2}\left\|x^{k, t}-x^{k, *}\right\|^{2}+\theta_{y}^{-1}\left\|y^{k, t}-y^{k, *}\right\|^{2} \\
& \geq \gamma_{x}^{-1}\left\|x^{k, t}-x^{k, *}\right\|^{2}+\gamma_{y}^{-1}\left\|y^{k, t}-y^{k, *}\right\|^{2}
\end{aligned}
$$

The latter inequality implies

$$
\begin{aligned}
\gamma_{x}^{-1}\left\|x^{k, t}-x^{k, *}\right\|^{2} & +\gamma_{y}^{-1}\left\|y^{k, t}-y^{k, *}\right\|^{2} \leq \\
& \leq \gamma_{x}\left\|a_{x}^{k}\left(x^{k, t}, y^{k, t}\right)+b_{x}^{k, t}\right\|^{2}+\gamma_{y}\left\|a_{y}^{k}\left(x^{k, t}, y^{k, t}\right)+b_{y}^{k, t}\right\|^{2}
\end{aligned}
$$

Further, we get

$$
\begin{aligned}
\gamma_{x}^{-1}\left\|x^{k,-1}-x^{k, *}\right\|^{2} & +\gamma_{y}^{-1}\left\|y^{k,-1}-y^{k, *}\right\|^{2} \\
& \leq 2 \gamma_{x}^{-1}\left\|x^{k,-1}-x^{k, t}\right\|^{2}+2 \gamma_{y}^{-1}\left\|y^{k,-1}-y^{k, t}\right\|^{2} \\
& +2 \gamma_{x}^{-1}\left\|x^{k, t}-x^{k, t}\right\|^{2}+2 \gamma_{y}^{-1}\left\|y^{k, t}-y^{k, t}\right\|^{2} \\
& \leq 2 \gamma_{x}^{-1}\left\|x^{k,-1}-x^{k, t}\right\|^{2}+2 \gamma_{y}^{-1}\left\|y^{k,-1}-y^{k, t}\right\|^{2} \\
& +2 \gamma_{x}\left\|a_{x}^{k}\left(x^{k, t}, y^{k, t}\right)+b_{x}^{k, t}\right\|^{2}+2 \gamma_{y}\left\|a_{y}^{k}\left(x^{k, t}, y^{k, t}\right)+b_{y}^{k, t}\right\|^{2} .
\end{aligned}
$$

Using Theorem 2 and the definition 27) of operators $A^{k}(u)$ and $B(u)$ we get the following inequality:

$$
\begin{aligned}
\gamma_{x}\left\|a_{x}^{k}\left(x^{k, t}, y^{k, t}\right)+b_{x}^{k, t}\right\|^{2} & +\gamma_{y}\left\|a_{y}^{k}\left(x^{k, t}, y^{k, t}\right)+b_{y}^{k, t}\right\|^{2} \\
& \leq \frac{288 M^{2}}{(t+1)^{2}}\left(\gamma_{x}^{-1}\left\|x^{k,-1}-x^{k, *}\right\|^{2}+\gamma_{y}^{-1}\left\|y^{k,-1}-y^{k, *}\right\|^{2}\right) \\
& \leq \frac{288 M^{2}}{(t+1)^{2}}\left(2 \gamma_{x}^{-1}\left\|x^{k,-1}-x^{k, t}\right\|^{2}+2 \gamma_{y}^{-1}\left\|y^{k,-1}-y^{k, t}\right\|^{2}\right) \\
& +\frac{288 M^{2}}{(t+1)^{2}}\left(2 \gamma_{x}\left\|a_{x}^{k}\left(x^{k, t}, y^{k, t}\right)+b_{x}^{k, t}\right\|^{2}+2 \gamma_{y}\left\|a_{y}^{k}\left(x^{k, t}, y^{k, t}\right)+b_{y}^{k, t}\right\|^{2}\right) \\
& =\frac{576 M^{2}}{(t+1)^{2}}\left(\gamma_{x}^{-1}\left\|x^{k,-1}-x^{k, t}\right\|^{2}+\gamma_{y}^{-1}\left\|y^{k,-1}-y^{k, t}\right\|^{2}\right) \\
& +\frac{576 M^{2}}{(t+1)^{2}}\left(\gamma_{x}\left\|a_{x}^{k}\left(x^{k, t}, y^{k, t}\right)+b_{x}^{k, t}\right\|^{2}+s \gamma_{y}\left\|a_{y}^{k}\left(x^{k, t}, y^{k, t}\right)+b_{y}^{k, t}\right\|^{2}\right)
\end{aligned}
$$

Now, we set $t=T$, where $T$ is defined by (33), and use the definition (30) of $M$. This implies.

$$
\begin{aligned}
\gamma_{x}\left\|a_{x}^{k}\left(x^{k, t}, y^{k, t}\right)+b_{x}^{k, t}\right\|^{2} & +\gamma_{y}\left\|a_{y}^{k}\left(x^{k, t}, y^{k, t}\right)+b_{y}^{k, t}\right\|^{2} \\
& \leq \frac{1}{2}\left(\gamma_{x}^{-1}\left\|x^{k,-1}-x^{k, t}\right\|^{2}+\gamma_{y}^{-1}\left\|y^{k,-1}-y^{k, t}\right\|^{2}\right) \\
& +\frac{1}{2}\left(\gamma_{x}\left\|a_{x}^{k}\left(x^{k, t}, y^{k, t}\right)+b_{x}^{k, t}\right\|^{2}+\gamma_{y}\left\|a_{y}^{k}\left(x^{k, t}, y^{k, t}\right)+b_{y}^{k, t}\right\|^{2}\right)
\end{aligned}
$$

After rearranging, we obtain condition (31) with $t=T$.
Now, one can observe that there are two possibilities: the while-loop of Algorithm 4 stops when $t=T$, otherwise it stops earlier. This concludes the proof.

## G Proof of Theorem 3

According to Theorem 1 , the following number of outer iterations of Algorithm 4 are required to find an $\epsilon$-accurate solution:

$$
\begin{equation*}
K=\mathcal{O}\left(\max \left\{\frac{1}{\alpha}, \frac{\alpha}{\theta_{y} \mu_{y}}\right\} \log \frac{1}{\epsilon}\right) \tag{49}
\end{equation*}
$$

Using the definition of $\alpha$ and $\theta_{y}$ we get

$$
\begin{equation*}
K=\mathcal{O}\left(\max \left\{1, \sqrt{\frac{\mu_{x}}{\mu_{y}}}\right\} \log \frac{1}{\epsilon}\right) \tag{50}
\end{equation*}
$$

According to Lemma 4 at most $T$ inner iterations are performed by Algorithm 4 at each outer iteration, where $T$ is given as

$$
\begin{equation*}
T=\mathcal{O}\left(\max \left\{\frac{L}{\mu_{x}}, \theta_{y} L\right\}\right) \tag{51}
\end{equation*}
$$

Using the definition of $\theta_{y}$ we get

$$
\begin{equation*}
T=\mathcal{O}\left(\frac{L}{\mu_{x}}\right) \tag{52}
\end{equation*}
$$

Now, we get the total number of iterations:

$$
\begin{equation*}
K \times T=\mathcal{O}\left(\max \left\{1, \sqrt{\frac{\mu_{x}}{\mu_{y}}}\right\} \log \frac{1}{\epsilon}\right) \times \mathcal{O}\left(\frac{L}{\mu_{x}}\right)=\mathcal{O}\left(\max \left\{\frac{L}{\mu_{x}}, \frac{L}{\sqrt{\mu_{x} \mu_{y}}}\right\} \log \frac{1}{\epsilon}\right) \tag{53}
\end{equation*}
$$

It remains to observe that Algorithm 4 performs $\mathcal{O}(1)$ gradient evaluations per iteration.


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    ${ }^{3}$ Most existing works on minimax optimization study the convex-concave case. However, this setting can be easily reduced to the strongly-convex-strongly-concave case via the regularization technique (Lin et al. 2020).

[^1]:    ${ }^{4}$ In contrast to smooth convex-concave minimax optimization, the answer to this question for smooth convex minimization was given by Nesterov (1983) several decades ago.

[^2]:    ${ }^{5}$ Recall that for a convex function $h(x)$, Fenchel conjugate is defined as $h^{*}(z)=\sup _{x}[\langle z, x\rangle-h(x)]$.

[^3]:    ${ }^{6}$ To be precise, Lemma 1 implies the relation $17 \Rightarrow 16$ rather than the equivalence 17$] \Leftrightarrow 16$. However, this is not an issue because we provide the intuition behind the algorithm development in this section. The rigorous proofs are postponed to the Appendix.

