

Recent theoretical advances in decentralized distributed convex optimization

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Abstract In the last few years, the theory of decentralized distributed convex optimization has made significant progress. The lower bounds on communications rounds and oracle calls have appeared, as well as methods that reach both of these bounds. In this paper, we focus on how these results can be explained based on optimal algorithms for the non-distributed setup. In particular, we provide our recent results that have not been published yet and that could be found in details only in arXiv preprints.

1 Introduction

In this work, we focus on the following convex optimization problem

$$\min_{x \in Q \subseteq \mathbb{R}^n} f(x) := \frac{1}{m} \sum_{i=1}^m f_i(x), \quad (1)$$

where the functions $\{f_i\}_{i=1}^m$ are convex and Q is a convex set. Such kind of problems arise in many machine learning applications [138] (e.g., empirical risk minimization) and statistical applications [145] (e.g., maximum likelihood estimation). To solve these problems, decentralized distributed methods are widely used (see [110, 33] and reference therein). This direction has gained popularity with the release of the book [13]. Many researchers (among which we especially note Angelia Nedich) have productively promoted distributed algorithms in the last 30 years. Due to the emergence of big data and the rapid growth of problem sizes, decentralized distributed methods have gained increased interest in the last decade. In this paper, we mainly focus on the last five years of theoretical advances, starting with the remarkable paper [8]. The authors of [8] introduce the lower complexity bounds for communication rounds required to achieve ε -accuracy solution x^N of (1) in the function value, i.e., $f(x^N) - \min_{x \in Q} f(x) \leq \varepsilon$.

Let us formulate the result of [8] (see also [142, 134, 136, 87, 159]) formally. Assume that we have some connected undirected graph (network) with m nodes. For each node i of this graph, we privately assign function f_i and suppose that the node i can calculate ∇f_i at some point x . At each communication round the nodes can communicate with their neighbors, i.e., send and receive a message with no more than $O(n)$ numbers. In the $O(R)$ neighborhood of a solution x^* of (1) (where $R = \|x^0 - x^*\|_2$ is the Euclidean distance between starting point x^0 and the solution x^* that corresponds to the minimum of this norm), we suppose that functions f_i 's are M -Lipschitz continuous (i.e., $\|\nabla f_i(x)\|_2 \leq M$) and L -Lipschitz smooth (i.e., $\|\nabla f_i(y) - \nabla f_i(x)\|_2 \leq L\|y - x\|_2$). The optimal bounds on the number of communications and the number of oracle calls per node are summarized in Table 1. Here and below $\tilde{O}(\cdot)$ means the same as $O(\cdot)$ up to a $\log(1/\varepsilon)$ factor, and $\tilde{O}(\sqrt{\chi})$ corresponds to the consensus time, that is the number of communication rounds required to reach the consensus in the considered network (more accurate definition of $\tilde{O}(\sqrt{\chi})$ is given in Sections 2, 3).

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In the last few years, algorithms have been developed that reach the lower bounds from Table 1. In Section 2, we consider one of such algorithms [131, 133] for the case when the functions f_i 's are smooth. This algorithm has the simplest nature among all known alternatives: this is a direct consensus-projection generalization of Nesterov's fast gradient method.

When communication networks vary from time to time (time-varying communication networks, see Section 2), we replace $\sqrt{\chi}$ by χ and we suppose that different f_i 's may have different constants of smoothness L_i .

The non-smooth case (when functions f_i 's are Lipschitz continuous) is studied in Section 3, where the results from [32, 51] are summarized. The approach is based on reformulation of the distributed decentralized problem as non-distributed convex optimization problem with affine constraints, which are further brought into the target function as a composite quadratic penalty. To solve this problem, Lan's sliding algorithm [86] can be used.

Table 1 Optimal bounds for communication rounds and deterministic oracle calls of ∇f_i per node

	f_i is μ -strongly convex and L -smooth	f_i is L -smooth	f_i is μ -strongly convex	
# communication rounds	$\tilde{O}\left(\sqrt{\frac{L}{\mu}\chi}\right)$	$\tilde{O}\left(\sqrt{\frac{LR^2}{\varepsilon}\chi}\right)$	$O\left(\sqrt{\frac{M^2}{\mu\varepsilon}\chi}\right)$	$O\left(\sqrt{\frac{M^2R^2}{\varepsilon^2}\chi}\right)$
# oracle calls of ∇f_i per node i	$\tilde{O}\left(\sqrt{\frac{L}{\mu}}\right)$	$O\left(\sqrt{\frac{LR^2}{\varepsilon}}\right)$	$O\left(\frac{M^2}{\mu\varepsilon}\right)$	$O\left(\frac{M^2R^2}{\varepsilon^2}\right)$

The same construction and estimates hold (see Table 2) in the non-smooth stochastic case, when instead of subgradients $\nabla f_i(x)$'s we have an access only to their unbiased estimates $\nabla f_i(x, \xi_i)$'s. We assume here that $\mathbb{E}\|\nabla f(x, \xi)\|_2^2 \leq M^2$ on a $O(R)$ neighborhood of x^* .

Table 2 Optimal bounds for communication rounds and stochastic oracle calls of $\nabla f_i(x, \xi_i)$ per node

	f_i is μ -strongly convex and L -smooth	f_i is L -smooth	f_i is μ -strongly convex	
# communication rounds	$\tilde{O}\left(\sqrt{\frac{L}{\mu}\chi}\right)$	$\tilde{O}\left(\sqrt{\frac{LR^2}{\varepsilon}\chi}\right)$	$O\left(\sqrt{\frac{M^2}{\mu\varepsilon}\chi}\right)$	$O\left(\sqrt{\frac{M^2R^2}{\varepsilon^2}\chi}\right)$
# oracle calls of $\nabla f_i(x, \xi_i)$ per node i	$\tilde{O}\left(\max\left\{\frac{\sigma^2}{m\mu\varepsilon}, \sqrt{\frac{L}{\mu}}\right\}\right)$	$O\left(\max\left\{\frac{\sigma^2R^2}{m\varepsilon^2}, \sqrt{\frac{LR^2}{\varepsilon}}\right\}\right)$	$O\left(\frac{M^2}{\mu\varepsilon}\right)$	$O\left(\frac{M^2R^2}{\varepsilon^2}\right)$

The smooth part of Table 2 describes the known lower bounds. There exist methods that are optimal only in one of the two mentioned criteria [32]: either in communication rounds or in oracle calls per node. The technique from [133] (also described in Section 2) combined with proper batch-size policy [36] allows to reach these lower bounds up to a logarithmic factor [130].

Section 3 also contains analogues of the results mentioned in Tables 1 and 2 for dual (stochastic) gradient type oracle. That is, instead of an access at each node to ∇f_i we have an access to the gradient of conjugated function ∇f_i^* [134, 159]. Such oracle appears in different applications, in particular, in Wasserstein barycenter problem [157, 37, 79, 34, 31].

In Section 4, we transfer the results mentioned above to gradient-free oracle assuming that we have an access only to f_i instead of ∇f_i . In this case, a trivial solution comes to mind: to restore the gradient from finite differences. Based on optimal gradient-type methods in smooth case, it is possible to build optimal gradient-free methods. But what is about non-smooth case? To the best of our knowledge, until recently, it was an open question. Based on [15], we provide an answer for this question (Section 4). To say more precisely, we transfer optimal gradient-free algorithms for non-smooth (stochastic two-points) convex optimization problems [140, 12] from non-distributed set up to decentralized distributed one. Here, as in Section 3, we also mainly use the penalty trick and the Lan's sliding.

It is worth to add several results to the list of recent advances collected in Tables 1, 2. The first result describes the case when $f_i(x) = \frac{1}{r} \sum_{j=1}^r f_i^j(x)$ in (1), $Q = \mathbb{R}^n$. All f_i^j are L -smooth and μ -strongly convex. In this case, the lower bounds were obtained in [59]. Optimal algorithms were proposed in [59, 96]. These algorithms require $\tilde{O}\left(\sqrt{\frac{L}{\mu}\chi}\right)$ communication rounds and $\tilde{O}\left(r + \sqrt{r\frac{L}{\mu}}\right)$ oracle calls (∇f_i^j calculations) per node. This is valuable result since in real machine learning applications the sum-type representation of f_i is typical.

Another way to use this representation is statistical similarity of f_i . The lower bound for communication rounds in deterministic smooth case was also obtained [8]. Roughly speaking, if the Hessians of the f_i 's are β -close in the 2-norm, then the lower bound for communication rounds will be $\tilde{\Omega}\left(\sqrt{\frac{\beta}{\mu}\chi}\right)$. Here $\tilde{\Omega}(\cdot)$ is the notation for lower bounds on the

growth rate hiding logarithms. For example, if $f_i(x) = \frac{1}{r} \sum_{j=1}^r f_i^j(x)$ and all f_i^j are μ -strongly convex we have $\beta \simeq \mu + \frac{\text{const}}{\sqrt{r}}$. In the decentralized distributed setup there is a gap between this lower bound and the optimal (non-accelerated) bound $\tilde{O}\left(\frac{\beta}{\mu}\chi\right)$ that can be achieved at the moment [152]. But for centralized distributed architectures with additional assumptions on f_i 's, partial acceleration is possible [60]. In the recent paper [154] the gap for decentralized optimization was closed by using distributed Catalyst.

Other group of results relate to very specific (but rather popular) centralized federated learning architectures [68]. According to mentioned above estimates, heterogeneous federated learning can be considered as a partial case (with $\chi = 1$) [70, 164, 75, 53]. In the paper [75], this was explained based on the analysis of unified decentralized SGD. Paper [75] also summarizes a lot of different distributed setups in one general approach. We partially try to use the generality from [75] in Section 2. To the best of our knowledge, it is an open question to accelerate all the results of [75]. Section 2 contains such an acceleration only in deterministic case.

Along with minimization, distributed saddle-point problems are an interesting venue of research [106, 162, 129, 103]. The basis for distributed solution of min-max problems are extragradient and Mirror-Prox methods [113]. Unlike minimization, where the optimal dependence of iteration complexity on function condition number is $\sqrt{\kappa}$, the lower complexity bound for saddle-point algorithms includes κ , and Nesterov acceleration does not improve classical non-accelerated methods. In decentralized case, the lower bound for number of communications is $O(\kappa\sqrt{\chi}\log(1/\varepsilon))$ [129]. Therefore, Nesterov acceleration technique is not needed for obtaining optimal methods for min-max problems both in classical and distributed optimization. But in particular cases (i.e. different constants of strong convexity and strong concavity) acceleration is possible due to distributed Catalyst [154] and [99, 45, 168, 155]. Note, also that lower bound and optimal decentralized algorithm for saddle-point problems with variance reduction was proposed in [17] (this paper develops the results of [59, 96]). Lower bound optimal decentralized algorithm for saddle-point problems with similarity was proposed in [20].

2 Decentralized Optimization of Smooth Convex Functions

Consider problem (1) and rewrite it in the following form:

$$\min_{X \in \mathbb{R}^{m \times n}} F(X) = \sum_{i=1}^m f_i(x_i) \quad \text{s.t. } x_1 = \dots = x_m \quad (2)$$

where $X = (x_1 \dots x_m)^\top$. Decentralized optimization problem is now reformulated as an optimization problem with linear constraints. The constraint set writes as $\mathbf{C} = \{x_1 = \dots = x_m\}$.

Functions f_i are stored on the nodes across the network, which is represented as an undirected graph $\mathcal{G} = (V, E)$. Every node has an access to the function and its first-order characteristics. Decentralized first-order methods use two types of steps – computational steps, i.e. performing local computations and communication steps, which is exchanging the information with neighbors. Alternating these two types of steps results in minimizing the objective while maintaining agents' vectors approximately equal.

We begin with an overview of how communication procedures are developed and analyzed. The iterative information exchange is referred to as *consensus* or *gossip* algorithms in the literature [22, 156, 166, 109].

2.1 Consensus algorithms

Let each agent in the network initially hold a vector x_i^0 and let the communication network be represented by a connected graph $\mathcal{G} = (V, E)$. The agents seek to find the average vector across the network, but their communication is restricted to sending and receiving information from their direct neighbors. In one communication round, every two nodes linked by an edge exchange their vector values. After that, agent i sums the received values with predefined coefficients m_{ij} , where j is the number of the corresponding neighbor. In other words, every node runs an update

$$x_i^{k+1} = [\mathbf{M}]_{ii}x_i^k + \sum_{(i,j) \in E} [\mathbf{M}]_{ij}x_j^k,$$

where $[\mathbf{M}]_{ij}$ are elements of the *mixing matrix* \mathbf{M} . The update at one communication step takes the form

$$X^{k+1} = \mathbf{M}X^k. \quad (3)$$

Under additional assumptions this iterative scheme converges to the average of initial vectors over network, i.e. to

$$\bar{X}^0 = \frac{1}{m} \mathbf{1} \mathbf{1}^\top X^0 = \frac{1}{m} \sum_{i=1}^m x_i^0.$$

Assumption 1 *Mixing matrix \mathbf{M} satisfies the following properties.*

- (Decentralized property) If $(i, j) \notin E$ and $i \neq j$, then $[\mathbf{M}]_{ij} = 0$. Otherwise $[\mathbf{M}]_{ij} > 0$.
- (Symmetry and double stochasticity) $\mathbf{M} \mathbf{1} = \mathbf{1}$ and $\mathbf{M} = \mathbf{M}^\top$.
- (Spectrum property) Denote $\lambda_2(\mathbf{M})$ the absolute value of second largest (in absolute value) eigenvalue of \mathbf{M} . Then $\lambda_2(\mathbf{M}) < 1$.

The choice of weights for mixing matrix is an interesting problem we do not address here (see [22] for details). A mixing matrix with Metropolis weights satisfies Assumption 1:

$$[\mathbf{M}]_{ij} = \begin{cases} 1/(1 + \max\{d_i, d_j\}) & \text{if } (i, j) \in E, \\ 0 & \text{if } (i, j) \notin E, \\ 1 - \sum_{m:(i,m) \in E} [\mathbf{M}]_{im} & \text{if } i = j, \end{cases}$$

where d_i denotes the degree of node i .

Several variations of Assumption 1 can be found in literature. In particular, in [101] the mixing matrix is not needed to be symmetric. Instead, it is assumed to be doubly stochastic and have a real spectrum. Moreover, the spectrum property in Assumption 1 implies that $\mathbf{1}$ is the only (up to a scaling factor) eigenvector corresponding to eigenvalue 1, i.e. $\ker(I - \mathbf{M}) = \text{span}(\mathbf{1})$.

Lemma 1. *For iterative consensus procedure (3) it holds*

$$\|X^k - \bar{X}^0\|_2 \leq (\lambda_2(\mathbf{M}))^k \|X^0 - \bar{X}^0\|_2.$$

Proof. Let $x \in \mathbb{R}^n$ and $\bar{x} = \frac{1}{m} \mathbf{1} \mathbf{1}^\top x$. First, note that $\mathbf{M}\bar{x} = \mathbf{M} \cdot \frac{1}{m} \mathbf{1} \mathbf{1}^\top x = \frac{1}{m} \mathbf{1} \mathbf{1}^\top x = \bar{x}$. It can be easily seen that $x - \bar{x} \in (\text{span}(\mathbf{1}))^\perp$ and $\mathbf{M}x - x \in (\text{span}(\mathbf{1}))^\top$:

$$\begin{aligned} \langle x - \bar{x}, \mathbf{1} \rangle &= \left\langle \left(I - \frac{1}{m} \mathbf{1} \mathbf{1}^\top \right) x, \mathbf{1} \right\rangle = \left\langle x, \left(I - \frac{1}{m} \mathbf{1} \mathbf{1}^\top \right) \mathbf{1} \right\rangle = 0, \\ \langle \mathbf{M}x - \bar{x}, \mathbf{1} \rangle &= \left\langle \left(\mathbf{M} - \frac{1}{m} \mathbf{1} \mathbf{1}^\top \right) x, \mathbf{1} \right\rangle = \left\langle x, \left(\mathbf{M} - \frac{1}{m} \mathbf{1} \mathbf{1}^\top \right) \mathbf{1} \right\rangle = 0. \end{aligned}$$

On subspace $(\text{span}(\mathbf{1}))^\top$ the largest eigenvalue of \mathbf{M} is $\lambda_2(\mathbf{M})$. We have

$$\|\mathbf{M}x - \bar{x}\|_2 = \|\mathbf{M}(x - \bar{x})\|_2 \leq \lambda_2(\mathbf{M}) \|x - \bar{x}\|_2.$$

Applying the derived fact to every column of X^k we get $\bar{X}^k = \bar{X}^0$ and $\|X^{k+1} - \bar{X}^0\|_2 \leq \lambda_2(\mathbf{M}) \|X^k - \bar{X}^0\|_2$ for every $k \geq 0$, which concludes the proof.

By Lemma 1, consensus scheme (3) requires $O\left(\frac{1}{1 - \lambda_2(\mathbf{M})} \log\left(\frac{1}{\varepsilon}\right)\right)$ iterations to achieve accuracy ε , i.e. to find arithmetic mean of vectors over the network with precision ε : $\|X^k - \bar{X}^0\|_2 \leq \varepsilon$.

Remark. If the graph changes with time, we associate a sequence of mixing matrices $\{\mathbf{M}^k\}_{k=0}^\infty$ with it. In the time-varying case, the consensus algorithm convergence rate is ruled by worst-case second largest eigenvalue, i.e. $\max_{k \geq 0} \lambda_2(\mathbf{M}^k)$.

Provided that each \mathbf{M}^k is symmetric, doubly stochastic and satisfies the decentralized property in Assumption 1, the number of communication rounds to reach consensus accuracy ε is $O\left(\frac{1}{1 - \max_{k \geq 0} \lambda_2(\mathbf{M}^k)} \log \frac{1}{\varepsilon}\right)$.

2.1.1 Quadratic optimization point of view

For a given undirected graph $\mathcal{G} = (V, E)$ introduce its Laplacian matrix

$$[\overline{W}]_{ij} = \begin{cases} -1, & \text{if } (i, j) \in E, \\ \deg(i), & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Laplacian matrix is positive semi-definite and for $X = (x_1 \dots x_m)^\top$ it holds $\overline{W}X = 0 \Leftrightarrow x_1 = \dots = x_m$. A more detailed discussion of Laplacian matrix and its applications is provided in Section 3.3. The consensus problem can be reformulated as

$$\min_{X \in \mathbb{R}^{m \times n}} g(X) := \frac{1}{2} \langle X, \overline{W}X \rangle. \quad (4)$$

Any matrix X^* with equal rows is a solution of Problem (4), and therefore the set of minimizers of Problem (4) is a linear subspace of form $\mathcal{X}^* = \{\mathbf{1}x^\top : x \in \mathbb{R}^n\}$. Denote $\lambda_{\max}(\overline{W})$ and $\lambda_{\min}^+(\overline{W})$ the largest and the smallest non-zero eigenvalues of \overline{W} , respectively. Then $g(X)$ has Lipschitz gradients with constant $\lambda_{\max}(\overline{W})$ and is strongly convex on $(\ker \overline{W})^\perp$ with modulus $\lambda_{\min}^+(\overline{W})$. Let non-accelerated gradient descent be run over function $g(X)$

$$X^{k+1} = X^k - \frac{1}{\lambda_{\max}(\overline{W})} \overline{W}X^k. \quad (5)$$

First, note that trajectory of method (5) stays in $X^0 + (\ker \overline{W})^\perp$. To verify this, consider $Z \in \ker \overline{W}$:

$$\begin{aligned} \langle \overline{W}X^k, Z \rangle &= \langle X^k, \overline{W}Z \rangle = 0 \Rightarrow \overline{W}X^k \in (\ker \overline{W})^\perp, \\ X^{N+1} - X^0 &= -\frac{1}{\lambda_{\max}(\overline{W})} \sum_{k=0}^N \overline{W}X^k \in (\ker \overline{W})^\perp. \end{aligned}$$

Method (5) converges to some point in \mathcal{X}^* . Since its trajectory lies in $X^0 + (\ker \overline{W})^\perp$, the limit point of $\{X^k\}_{k=0}^\infty$ is the projection of X^0 onto $\ker \overline{W}$, i.e. to \overline{X}^0 . The algorithm requires $O\left(\frac{\lambda_{\max}(\overline{W})}{\lambda_{\min}^+(\overline{W})} \log\left(\frac{1}{\varepsilon}\right)\right)$ iterations to reach accuracy ε .

In order to establish the connection between gradient descent for problem (4) and consensus algorithm (3), introduce $\mathbf{M} = I - \frac{\overline{W}}{\lambda_{\max}(\overline{W})}$. Matrix \mathbf{M} satisfies Assumption 1, and update rule (5) then rewrites as

$$X^{k+1} = X^k - \frac{\overline{W}}{\lambda_{\max}(\overline{W})} X^k = \left(I - \frac{\overline{W}}{\lambda_{\max}(\overline{W})}\right) X^k = \mathbf{M}X^k.$$

Therefore, gradient descent on $g(X)$ with constant step-size $\frac{1}{\lambda_{\max}(\overline{W})}$ is equivalent to non-accelerated consensus algorithm (3). Moreover, the iteration complexities coincide, since $\lambda_2(\mathbf{M}) = 1 - \frac{\lambda_{\min}^+(\overline{W})}{\lambda_{\max}(\overline{W})}$ and therefore $O\left(\frac{\lambda_{\max}(\overline{W})}{\lambda_{\min}^+(\overline{W})} \log\left(\frac{1}{\varepsilon}\right)\right) = O\left(\frac{1}{1-\lambda_2(\mathbf{M})} \log\left(\frac{1}{\varepsilon}\right)\right)$.

We note that the same gradient descent analogy holds for time-varying networks. Given a sequence of connected undirected graphs $\{\mathcal{G}^k\}_{k=0}^\infty$, consider a sequence of corresponding Laplacians $\{\overline{W}^k\}_{k=0}^\infty$. The consensus algorithm may be interpreted as a gradient descent on a time-varying quadratic function $\{g^k(X)\}_{k=0}^\infty$, where $g^k(X)$ is defined as

$$g^k(X) = \frac{1}{2} \langle X, \overline{W}^k X \rangle. \quad (6)$$

All $g^k(X)$ have a common set of minimizers $\{\mathbf{1}x^\top : x \in \mathbb{R}^d\}$. The worst-case Lipschitz constant over time is $\max_{k \geq 0} \lambda_{\max}(\overline{W}^k)$, and consensus iteration writes similar to (5) with $\lambda_{\max}(\overline{W})$ replaced by $\max_{k \geq 0} \lambda_{\max}(\overline{W}^k)$. The convergence guarantees of gradient descent over a time varying function $\{g^k(X)\}_{k=0}^\infty$ do not break since the Lyapunov function for non-accelerated gradient dynamics is the squared distance to the solution set, i.e. it does not depend on the minimization objective. Therefore,

non-accelerated gradient descent decreases the Lyapunov function at each step and is robust to changes in the objective function. The number of communication rounds to reach consensus accuracy ε is $O\left(\frac{\max_{k \geq 0} \lambda_{\max}(\overline{W}^k)}{\max_{k \geq 0} \lambda_{\min}^+(\overline{W}^k)} \log \frac{1}{\varepsilon}\right)$.

In order to obtain a better dependence on $\frac{\lambda_{\max}(\overline{W})}{\lambda_{\min}^+(\overline{W})}$, Nesterov acceleration [115] may be employed. Consider Nesterov accelerated method for strongly convex objectives

$$\beta = \frac{\sqrt{\lambda_{\max}(\overline{W})} - \sqrt{\lambda_{\min}^+(\overline{W})}}{\sqrt{\lambda_{\max}(\overline{W})} + \sqrt{\lambda_{\min}^+(\overline{W})}}, \quad (7a)$$

$$Y^k = X^k + \beta(X^k - X^{k-1}), \quad (7b)$$

$$X^{k+1} = Y^k - \frac{\overline{W}}{\lambda_{\max}(\overline{W})} Y^k. \quad (7c)$$

Analogously to non-accelerated scheme (5), the trajectory of accelerated Nesterov method lies in $X^0 + (\ker \overline{W})^\perp$. This can be easily seen by induction:

$$\begin{aligned} Y^k - X^0 &= (X^k - X^0) + \beta((X^k - X^0) - (X^{k-1} - X^0)) \in (\ker \overline{W})^\perp, \\ X^{k+1} - X^0 &= \underbrace{(Y^k - Y^0)}_{\in (\ker \overline{W})^\perp} - \frac{1}{\lambda_{\max}(\overline{W})} \underbrace{\overline{W} Y^k}_{\in \text{Im } \overline{W} = (\ker \overline{W})^\perp} \in (\ker \overline{W})^\perp. \end{aligned}$$

Therefore, accelerated scheme converges to the projection of X^0 onto $\ker \overline{W}$, which is \overline{X}^0 , i.e. a matrix which rows are arithmetic averages of X^0 .

Note that acceleration is not attainable over time-varying graphs. Imagine we run Nesterov gradient method over a time-varying objective $\{g^k(X)\}_{k=0}^\infty$ defined in (6). The potential function for accelerated gradient dynamics [10] includes the objective function. Since the objective function changes, the potential function is also time-dependent and this fact breaks the prove of convergence result. A formal proof of why acceleration is impossible over time-varying networks that stay connected and undirected at each iteration is provided in [76].

2.1.2 Chebyshev acceleration

As shown in Section 2.1.1, the convergence of consensus algorithm depends on the condition number $\chi = \frac{\lambda_{\max}(\overline{W})}{\lambda_{\min}^+(\overline{W})}$. The factor χ also appears in upper complexity bounds for decentralized algorithms [134] and represents the measure of the communication graph connectivity. There exists a technique called *Chebyshev acceleration* that enhances the dependence on χ by replacing the communication matrix \overline{W} with a Chebyshev polynomial $P_K(\overline{W})$. The structure of the polynomial ensures that the condition number of $P_K(\overline{W})$ is $O(1)$ whence its power is $K = \lfloor \sqrt{\chi} \rfloor$. In other words, multiplication by $P_K(\overline{W})$ requires $\lfloor \sqrt{\chi} \rfloor$ communication rounds, but the condition number is reduced from χ to $O(1)$. A multiple $P_K(\overline{W})X$ is computed via an iterative consensus-based process. Introduce $c_2 = \frac{\chi+1}{\chi-1}$, $a_0 = 1$, $a_1 = c_2$, $c_3 = \frac{2}{\lambda_{\max}(\overline{W}) + \lambda_{\min}^+(\overline{W})}$, $X^0 = X$, $X^1 = c_2(I - c_3 \overline{W})X$, for $t = 1, \dots, K-1$ do

$$a_{t+1} = 2c_2 a_t - a_{t-1}, \quad X^{t+1} = 2c_2(I - c_3 \overline{W})X^t - X^{t-1}$$

and return $X^0 - X^K/a_K$.

Summing up, a consensus algorithm of form $X^{k+1} = (I - P_K(\overline{W})/\lambda_{\max}(P_K(\overline{W})))X^k$ requires a total of $O(\sqrt{\chi} \log 1/\varepsilon)$ communications to achieve accuracy ε . This complexity is better than $O(\chi \log 1/\varepsilon)$ that corresponds to standard consensus algorithm $X^{k+1} = (I - \overline{W}/\lambda_{\max}(\overline{W}))X^k$. Chebyshev acceleration was used to obtain first optimal decentralized methods in [134].

2.1.3 Summary

In this section, we covered consensus algorithms over time-static and time-varying undirected graphs.

Firstly, let us cover the time-static networks. Different types of matrices may correspond to the graph: mixing matrix \mathbf{M} or Laplacian matrix $\overline{\mathbf{W}}$. The key difference is that mixing matrix is doubly stochastic, i.e. $\mathbf{M}\mathbf{1} = \mathbf{1}$ and $\mathbf{1}^\top \mathbf{M} = \mathbf{1}^\top$, while the Laplacian has a null-space property $\overline{\mathbf{W}}\mathbf{1} = \mathbf{0}$. Consensus step based on mixing matrix is just multiplication by \mathbf{M} ; therefore, it does not require additional knowledge of mixing matrix spectrum. Concerning Laplacian $\overline{\mathbf{W}}$, we can either build a mixing matrix $I - \overline{\mathbf{W}}/\lambda_{\max}(\overline{\mathbf{W}})$ or use a quadratic minimization approach (see Section 2.1.1). In both cases, knowledge of $\lambda_{\max}(\overline{\mathbf{W}})$ is required.

Secondly, acceleration techniques are also applicable to consensus iteration schemes, but only in the *time-static* case. We can use Chebyshev acceleration covered in Section 2.1.2 or employ an accelerated Nesterov method to a quadratic problem (Section 2.1.1). In both cases, we improve the iteration complexity from $O(\chi \log(\frac{1}{\varepsilon}))$ to $O(\sqrt{\chi} \log(\frac{1}{\varepsilon}))$. However, acceleration is not attainable in the time-varying case. When the graph changes, we can only run non-accelerated consensus.

2.2 Main assumptions on objective functions

In this section, we introduce basic assumptions on the functions locally held by computational entities in the network.

Assumption 2 For every $i = 1, \dots, m$, function f_i is differentiable, convex and L_i -smooth ($L_i > 0$).

Assumption 3 For every $i = 1, \dots, m$, function f_i is μ_i -strongly convex ($\mu_i > 0$).

Under Assumptions 2 and 3 for any $x_i, y_i \in \mathbb{R}^m$ for $i = 1, \dots, m$ it holds

$$\frac{\mu_i}{2} \|y_i - x_i\|_2^2 \leq f_i(y_i) - f_i(x_i) - \langle \nabla f(x_i), y_i - x_i \rangle \leq \frac{L_i}{2} \|y_i - x_i\|_2^2.$$

Summing the above inequality on $i = 1, \dots, m$ we obtain

$$\frac{\min_i \mu_i}{2} \|Y - X\|_2^2 \leq \sum_{i=1}^m \frac{\mu_i}{2} \|y_i - x_i\|_2^2 \leq F(Y) - F(X) - \langle \nabla F(X), Y - X \rangle \leq \sum_{i=1}^m \frac{L_i}{2} \|y_i - x_i\|_2^2 \leq \frac{\max_i L_i}{2} \|Y - X\|_2^2.$$

On the other hand, given that $X, Y \in \mathbf{C}$, i.e. $x_1 = \dots = x_m$, $y_1 = \dots = y_m$, we have

$$\frac{1}{2m} \sum_{i=1}^m \mu_i \|Y - X\|_2^2 \leq F(Y) - F(X) - \langle \nabla F(X), Y - X \rangle \leq \frac{1}{2m} \sum_{i=1}^m L_i \|Y - X\|_2^2.$$

Therefore, $F(X)$ has different strong convexity and smoothness constants on $\mathbb{R}^{m \times n}$ and \mathbf{C} . Following the definitions in [134], we introduce

- (local constants) $F(X)$ is μ_l -strongly convex and L_l -smooth on $\mathbb{R}^{m \times d}$, where $\mu_l = \min_i \mu_i$, $L_l = \max_i L_i$.
- (global constants) $F(X)$ is μ_g -strongly convex and L_g -smooth on \mathbf{C} , where $\mu_g = \frac{1}{m} \sum_{i=1}^m \mu_i$, $L_g = \frac{1}{m} \sum_{i=1}^m L_i$.

Note that local smoothness and convexity constants may be significantly worse than global, i.e. $L_l \gg L_g$, $\mu_l \ll \mu_g$ (see [134] for details). We denote

$$\kappa_l = \frac{L_l}{\mu_l}, \quad \kappa_g = \frac{L_g}{\mu_g}. \quad (8)$$

the local and global condition numbers, respectively. A trick proposed in [134] allows to improve the local condition number by slightly changing functions f_i . Namely, introduce $\hat{f}_i(x) = f_i(x) - \frac{\mu_i - \mu_g}{2} \|x\|_2^2$ instead of f_i . Then the local condition number writes as

$$\hat{\kappa}_l = \frac{\max_k (L_k - \mu_k)}{\mu_g} + 1.$$

2.3 Distributed gradient descent

Distributed gradient methods alternate taking optimization updates and information exchange steps. One (synchronized) communication round can be represented as a multiplication by a mixing matrix compatible with the graph topology. One of the first distributed gradient dynamics studied in the literature [112, 172] uses a time-static mixing matrix and writes as

$$x_i^{k+1} = \sum_{j=1}^m [\mathbf{M}]_{ij} x_j^k - \alpha^k \nabla f_i(x_i^k).$$

Using the notion of $X = (x_1 \dots x_m)^\top$ the above update rule takes the form

$$X^{k+1} = \mathbf{M}X^k - \alpha \nabla F(X^k), \quad (9)$$

which is a combination of two step types: gradient step with constant step-size α and communication round with mixing matrix \mathbf{M}^k . In [172] the authors showed that function residual $f(\bar{X}^k) - f(x^*)$ in iterative scheme (9) decreases at $O(1/k)$ rate until reaching $O(\alpha)$ -neighborhood of solution.

Method (9) does not find an exact solution in the general case. We follow the arguments in [142] to illustrate this fact. First, note that \bar{X} is a solution of (2) if and only if two following conditions hold.

1. (Consensus) $\bar{X} = \mathbf{M}\bar{X}$.
2. (Optimality) $\mathbf{1}^\top \nabla F(\bar{X}) = 0$.

Let X^∞ be a limit point of (9). Then

$$X^\infty = \mathbf{M}X^\infty - \alpha \nabla F(X^\infty).$$

Consensus condition yields $X^\infty = \mathbf{M}X^\infty$, i.e. X^∞ has identical rows $[X]_i^\infty = x^\infty$. Therefore, $\nabla F(X^\infty) = 0$, which means $\nabla f_1(x^\infty) = \dots = \nabla f_m(x^\infty) = 0$. Consequently, x^∞ is a common minimizer of every f_i , which is not a realistic case.

Method (9) is a basic distributed first-order method. Its different variations include feasible point algorithms [92] and sub-gradient methods [112] (actually, the latter work initially proposed scheme (9)). Extensions to stochastic objectives and stochastic mixing matrices have been addressed in [75, 2, 100].

2.4 EXTRA

Distributed gradient descent (9) is unable to converge to the exact minimum of (2), which is the major drawback of the method. An exact decentralized first-order algorithm EXTRA was proposed in [142]. The approach of [142] is based on using two different mixing matrices. Namely, consider two consequent updates of type (9).

$$X^{k+2} = \mathbf{M}X^{k+1} - \alpha \nabla F(X^{k+1}), \quad (10)$$

$$X^{k+1} = \tilde{\mathbf{M}}X^k - \alpha \nabla F(X^k) \quad (11)$$

where $\tilde{\mathbf{M}}$ is mixing matrix, i.e. $\tilde{\mathbf{M}} = (\mathbf{M} + I)/2$ as proposed in [142]. Subtracting (11) from (10) yields

$$X^{k+2} - X^{k+1} = \mathbf{M}X^{k+1} - \tilde{\mathbf{M}}X^k - \alpha [\nabla F(X^{k+1}) - \nabla F(X^k)] \quad (12)$$

thus leading to an algorithm

Algorithm 1 EXTRA

Input: Step-size $\alpha > 0$.

$$X^1 = \mathbf{M}X^0 - \alpha \nabla F(X^0)$$

for $k = 0, 1, \dots$ **do**

$$X^{k+2} = (I + \mathbf{M})X^{k+1} - \tilde{\mathbf{M}}X^k - \alpha [\nabla F(X^{k+1}) - \nabla F(X^k)]$$

end for

Let X^∞ be a limit point of iterate sequence $\{X^k\}_{k=0}^\infty$ generated by (10), (11). Then

$$\begin{aligned} X^\infty - X^\infty &= \mathbf{M}X^\infty - \tilde{\mathbf{M}}X^\infty - \alpha[\nabla F(X^\infty) - \nabla F(X^\infty)], \\ (\mathbf{M} - \tilde{\mathbf{M}})X^\infty &= \frac{1}{2}(\mathbf{M}X^\infty - X^\infty) = 0. \end{aligned}$$

The last equality means that X^∞ is consensual, i.e. its rows are equal. On the other hand, rearranging the terms in (12) and taking into account that $X^1 = \mathbf{M}X^0 - \alpha\nabla F(X^0)$ gives

$$X^{k+2} = \tilde{\mathbf{M}}X^{k+1} - \alpha\nabla F(X^{k+1}) + \sum_{t=0}^{k+1} (\mathbf{M} - \tilde{\mathbf{M}})X^t.$$

Multiplying by $\mathbf{1}^\top$ from the left yields

$$\begin{aligned} \mathbf{1}^\top X^{k+2} &= \mathbf{1}^\top \tilde{\mathbf{M}}X^{k+1} - \alpha\mathbf{1}^\top \nabla F(X^{k+1}) + \sum_{t=0}^{k+1} \mathbf{1}^\top (\mathbf{M} - \tilde{\mathbf{M}})X^t \\ &= \mathbf{1}^\top \tilde{\mathbf{M}}X^{k+1} - \alpha\mathbf{1}^\top \nabla F(X^{k+1}) \end{aligned}$$

and taking the limit over $k \rightarrow \infty$ we obtain

$$\mathbf{1}^\top \nabla F(X^\infty) = 0$$

which is the optimality condition for point X^∞ . Therefore, a limit point of $\{X^k\}_{k=0}^\infty$ generated by Algorithm 1 is both consensual and optimal, i.e. is a solution of (2).

In the original paper [142] Algorithm 1 was proved to converge at a $O(1/k)$ rate for L -smooth objectives and achieve a geometric rate $O(C^{-k})$ (where $C < 1$ is some constant) for strongly convex smooth objectives. In [94] explicit dependencies on graph topology were established. Namely, EXTRA requires

$$\begin{aligned} O\left(\left(\frac{L_l}{\mu_l} + \chi\right) \log \frac{(LR^2 + \tilde{M}^2/L)\chi}{\varepsilon}\right) & \quad \text{iterations for strongly convex smooth objectives,} \\ O\left(\left(\frac{L_l}{\varepsilon} + \chi\right) \log((LR^2 + \tilde{M}^2/L)\chi)\right) & \quad \text{iterations for (non-strongly) convex smooth objectives,} \end{aligned}$$

where

$$\chi = \frac{1}{1 - \lambda_2(\mathbf{M})}, \quad (13)$$

$\lambda_2(\mathbf{M})$ denotes the second largest eigenvalue of mixing matrix \mathbf{M} , $\|X^0 - X^*\|_2^2 \leq mR^2$, $\|X^*\|_2^2 \leq mR^2$, $\|\nabla f(X^*)\|_2^2 \leq m\tilde{M}^2$. The term χ characterizes graph connectivity. A similar term, also referred to as graph condition number, is used in Section 3.3 for graph Laplacian matrix. Graph condition numbers based on mixing matrix and Laplacian have the same meaning, as discussed in Section 2.1.

2.5 Accelerated decentralized algorithms

Performance of decentralized gradient methods typically depends on function (local or global) condition number κ and graph condition number χ defined in (13). For non-accelerated dynamics [172, 142] complexity bounds include κ and χ . Improving dependencies to $\sqrt{\kappa}$ and $\sqrt{\chi}$ is an important direction of research in distributed optimization. This can be done by applying direct Nesterov acceleration [115] or by employing meta-acceleration techniques such as Catalyst [98]. The two major approaches studied in the literature are primal and dual algorithms.

Dual methods are based on a reformulation of problem (1) using a Laplacian matrix induced by the communication network. This reformulation is discussed in Section 3.3 in more details. The basic idea behind dual approach is to run first-order methods on a dual problem to (2). Every gradient step on the dual is equivalent to one communication round and one local gradient step taken by every node in the network. In [134], algorithms using Chebyshev acceleration that achieve $O(\sqrt{\kappa\chi} \log(1/\varepsilon))$ communication complexity are proposed. On the other hand, lower complexity bound for deterministic methods over strongly convex smooth objectives is $\Omega(\sqrt{\kappa\chi} \log(1/\varepsilon))$, as shown in [134].

In dual approach, one may run non-distributed accelerated schemes on dual problem and obtain accelerated complexity bounds, i.e. $\sqrt{\kappa\chi}$. For primal-only methods this is not the case, and primal algorithms have to alternate optimization and consensus steps in a proper way and employ specific techniques such as gradient tracking. A direct distributed scheme for Nesterov accelerated method was proposed in [125].

Algorithm 2 Accelerated Distributed Nesterov Method

Input: Starting points $X^0 = Y^0 = V^0$, $S^0 = \nabla F(X^0)$, step-size $\eta > 0$, momentum term $\alpha = \sqrt{\mu_l \eta}$

for $k = 0, 1, \dots$ **do**
 $X^{k+1} = \mathbf{M}Y^k - \eta S^k$
 $V^{k+1} = (1 - \alpha)\mathbf{M}V^k + \alpha\mathbf{M}Y^k - \frac{\eta}{\alpha}S^k$
 $Y^{k+1} = \frac{X^{k+1} + \alpha V^{k+1}}{1 + \alpha}$
 $S^{k+1} = \mathbf{M}S^k + \nabla F(Y^{k+1}) - \nabla F(Y^k)$
end for

In Algorithm 2 quantity S^{k+1} stands for a gradient estimator. The information about the gradients held by different agents is diffused through the network via consensus steps, i.e. $\mathbf{M}S^k$ multiplication. Every node stores one row of S^k which approximates the average gradient over the nodes in network:

$$s_i^k \approx \frac{1}{m} \sum_{k=1}^m \nabla f_i(y_i^k).$$

This technique is referred to as *gradient tracking* and is employed in several primal decentralized methods [125, 111, 169, 74, 3, 124, 123].

Algorithm 2 requires $O(\chi^{3/2} \kappa_l^{5/7} \log(1/\varepsilon))$ computation and communication steps to achieve accuracy ε , which does not match optimal bounds. EXTRA acceleration via Catalyst envelope [94] requires $O(\sqrt{\kappa_l \chi} \log \chi \log(1/\varepsilon))$ iterations for smooth strongly convex objectives. Recently a new method Mudag which unifies gradient tracking, Nesterov acceleration and multi-step consensus procedures was proposed in [169]. It has

$$\begin{aligned} &O\left(\sqrt{\kappa_g} \log\left(\frac{1}{\varepsilon}\right)\right) && \text{computation complexity and} \\ &O\left(\sqrt{\kappa_g \chi} \log\left(\frac{L_l}{L_g} \kappa_g\right) \log\left(\frac{1}{\varepsilon}\right)\right) && \text{communication complexity.} \end{aligned}$$

Mudag reaches optimal computation complexity and optimal communication complexity up to $\log\left(\frac{L_l}{L_g} \kappa_g\right)$ term. A valuable feature of the method is that it has dependencies on global condition number κ_l instead of local κ_g . In the general case, global condition number may be significantly better. A proximal version of Mudag method for composite optimization is studied in [170]. The method in [170] requires an optimal $O(\sqrt{\kappa_g \chi} \log(1/\varepsilon))$ number of computations and matches the lower communication complexity bound up to a logarithmic factor. Global condition number is also utilized in paper [133] where an inexact oracle framework [28, 29] for decentralized optimization is studied. The latter work is discussed in more details in Section 2.7. Finally, in [77] authors proposed a primal-only method OPAPC which reaches both optimal computation and communication complexities (up to replacing κ_g with κ_l).

Chebyshev acceleration is widely used to obtain optimal decentralized algorithms. For example, in [77] the authors propose method APAPC, which has $O\left(\left(\sqrt{\frac{L}{\mu}} \chi + \chi\right) \log \frac{1}{\varepsilon}\right)$ communication and computational complexities. After that, the authors replace Laplacian $\overline{\mathbf{W}}$ with a Chebyshev polynomial $P_K(\overline{\mathbf{W}})$, which results in $\chi(P_K(\overline{\mathbf{W}})) = O(1)$, but every communication round costs $O(\sqrt{\chi})$ communication rounds. Therefore, APAPC is modified to a new method OPAPC, which has $O\left(\sqrt{\frac{L}{\mu}} \log \frac{1}{\varepsilon}\right)$ oracle per node complexity and $O\left(\sqrt{\frac{L}{\mu}} \chi \log \frac{1}{\varepsilon}\right)$ communication complexity. In this particular case, Chebyshev acceleration not only allows to achieve optimal complexity bounds, but also separates oracle and communication complexities of the algorithm.

Paper [143] proposed OGT, a method based on *loopless* Chebyshev acceleration scheme. On the contrary to classical Chebyshev acceleration (used i.e. in OPAPC [77]), the loopless technique does not require multiple communication steps at each iteration. OGT requires $O\left(\sqrt{\frac{L}{\mu}} \log \frac{1}{\varepsilon}\right)$ oracle calls at each node and $O\left(\sqrt{\frac{L}{\mu}} \chi \log \frac{1}{\varepsilon}\right)$ communication steps, which meets the lower bounds.

Moreover, a recent work [144] showed that Nesterov acceleration can also be applied for distributed optimization over directed graphs. Their algorithm APD has communication complexity $\sim 1/\sqrt{\varepsilon}$ for non-strongly convex objectives and

APD-SC has $\sim \sqrt{L/\mu} \log(1/\varepsilon)$ complexity for strongly convex tasks. As stated in [143], in the case of undirected graphs the explicit dependence on network characteristics is attained: the complexity of APD-SC writes as $O\left(\sqrt{\frac{L}{\mu}} \chi^{3/2} \log \frac{1}{\varepsilon}\right)$.

2.6 Time-varying networks

In the time-varying case, the communication network changes from time to time. In practice this changes are typically caused by malfunctions such as loss of connection between the agents. The network is represented as a sequence of undirected communication graphs $\{\mathcal{G}^k = (V, E^k)\}_{k=0}^{\infty}$, and every graph \mathcal{G}^k is associated with a mixing matrix \mathbf{M}^k . The algorithms capable of working over time-varying graphs must be robust to sudden network changes. A linearly convergent method DIGing was proposed in [111].

Algorithm 3 DIGing

Input: Step-size $\alpha > 0$, starting iterate $X^0, Y^0 = \nabla F(X^0)$

for $k = 0, 1, \dots$ **do**
 $X^{k+1} = \mathbf{M}^k X^k - \alpha Y^k$
 $Y^{k+1} = \mathbf{M}^k Y^k + \nabla F(X^{k+1}) - \nabla F(X^k)$
end for

DIGing incorporates a gradient-tracking scheme and achieves linear convergence under realistic assumptions such as B -connectivity (i.e. a union of any B consequent graphs is connected). In [151] authors propose an algorithm which utilizes specific convex surrogates of local functions and local functions similarity in order to enhance convergence speed. In [95] a gradient-tracking technique combined with Nesterov acceleration was employed to construct an accelerated method AccGT over time-varying B -connected networks.

Another class of time-varying networks are graphs that stay connected at each iteration. For this type of problems, denote Laplacian at k -th iteration \overline{W}^k and define condition number $\chi_{tw} = \frac{\max_k \lambda_{\max}(\overline{W}^k)}{\min_k \lambda_{\min}^+(\overline{W}^k)}$. The lower bounds for this class of problems are $O(\sqrt{\kappa_l} \log(1/\varepsilon))$ for the number of (local) computations and $O(\chi \sqrt{\kappa_l} \log(1/\varepsilon))$ for the number of communications [76]. AccGT [95] and ADOM+ [76] are optimal algorithms using primal oracle, and ADOM [78] is an optimal dual method.

2.7 Inexact oracle point of view

In [133] the authors study an algorithm which alternates making gradient updates and running multi-step communication procedures. Introduce

$$\overline{X} = \frac{1}{m} \mathbf{1} \mathbf{1}^\top X = \Pi_{\mathcal{C}}(X) = (\overline{x} \dots \overline{x})^\top, \text{ where } \overline{x} = \frac{1}{m} \sum_{i=1}^m x_i \text{ and } \mathbf{1} = (1 \dots 1)^\top.$$

Also define an average gradient over nodes $\overline{\nabla F}(X) = 1/m \sum_{i=1}^m \nabla f_i(x_i)$. Consider a projection gradient method with trajectory lying in \mathcal{C}

$$\overline{X}_{k+1} = \overline{X}_k - \beta \overline{\nabla F}(\overline{X}_k).$$

In a centralized scenario, the computational network is endowed with a master agent, which communicates with all agents in the network. The master node is able to collect vectors x_i from every node in the network and compute a precise average \overline{x} . In decentralized case the master agent is not available, and therefore nodes are only able to compute an approximate average using consensus procedures. The network is allowed to change with time and the sequence of corresponding mixing matrices is restricted to the following assumption.

Assumption 4 *Mixing matrix sequence $\{\mathbf{M}^k\}_{k=0}^{\infty}$ satisfies the following properties.*

- (Decentralized property) $(i, j) \notin E_k \Rightarrow [\mathbf{M}^k]_{ij} = 0$.

- (Double stochasticity) $\mathbf{M}^k \mathbf{1} = \mathbf{1}$, $\mathbf{1}^\top \mathbf{M}^k = \mathbf{1}^\top$.
- (Contraction property) There exist $\tau \in \mathbb{Z}_{++}$ and $\lambda \in (0, 1)$ such that for every $k \geq \tau - 1$ it holds

$$\left\| \mathbf{M}_\tau^k X - \bar{X} \right\|_2 \leq (1 - \lambda) \|X - \bar{X}\|_2,$$

where $\mathbf{M}_\tau^k = \mathbf{M}^k \dots \mathbf{M}^{k-\tau+1}$.

Algorithm 4 Consensus

Input: Initial $X^0 \in \mathbf{C}$, number of iterations T .
for $t = 1, \dots, T$ **do**
 $X^{t+1} = \mathbf{M}^t X^t$
end for

Algorithm 5 Decentralized AGD with consensus subroutine

Input: Initial guess $X^0 \in \mathbf{C}$, constants $L, \mu > 0$, $U^0 = X^0$, $\alpha^0 = \mathbf{M}^0 = 0$
 1: **for** $k = 0, 1, 2, \dots$ **do**
 2: Find α^{k+1} as the greater root of $(A^k + \alpha^{k+1})(1 + A^k \mu) = L(\alpha^{k+1})^2$
 3: $A^{k+1} = A^k + \alpha^{k+1}$
 4: $Y^{k+1} = \frac{\alpha^{k+1} U^k + A^k X^k}{A^{k+1}}$
 5: $V^{k+1} = \frac{\mu Y^{k+1} + (1 + A^k \mu) U^k}{1 + A^k \mu + \mu} - \frac{\alpha^{k+1}}{1 + A^k \mu + \mu} \nabla F(Y^{k+1})$
 6: $U^{k+1} = \text{Consensus}(V^{k+1}, T^k)$
 7: $X^{k+1} = \frac{\alpha^{k+1} U^{k+1} + A^k X^k}{A^{k+1}}$
 8: **end for**

2.7.1 Inexact oracle construction

Trajectory of Algorithm 5 lies in the neighborhood of constraint set \mathbf{C} . It is analyzed in [133] (based on the technique developed in [131]) using the notation of inexact oracle [29, 28]. Algorithms of this type have been analyzed in time-static case [63] using inexact oracle notation, as well. Let $h(x)$ be a convex function defined on a convex set $Q \subseteq \mathbb{R}^m$. For $\delta > 0$, $L > \mu > 0$, a pair $(h_{\delta, L, \mu}(x), s_{\delta, L, \mu}(x))$ is called a (δ, L, μ) -model of $h(x)$ at point $x \in Q$ if for all $y \in Q$ it holds

$$\frac{\mu}{2} \|y - x\|_2^2 \leq h(y) - (h_{\delta, L, \mu}(x) + \langle s_{\delta, L, \mu}(x), y - x \rangle) \leq \frac{L}{2} \|y - x\|_2^2 + \delta. \quad (14)$$

The inexactness originates from computation of gradient at a point in neighborhood of \mathbf{C} . The next lemma identifies the size of neighborhood and describes the inexact oracle construction.

Lemma 2. *Define*

$$\begin{aligned} \delta &= \frac{1}{2n} \left(\frac{L_l^2}{L_g} + \frac{2L_l^2}{\mu_g} + L_l - \mu_l \right) \delta', \\ f_{\delta, L, \mu}(\bar{x}, X) &= \frac{1}{n} \left[F(X) + \langle \nabla F(X), \bar{X} - X \rangle + \frac{1}{2} \left(\mu_l - \frac{2L_l^2}{\mu_g} \right) \|\bar{X} - X\|_2^2 \right], \\ g_{\delta, L, \mu}(\bar{x}, X) &= \frac{1}{n} \sum_{i=1}^n \nabla f_i(x_i). \end{aligned} \quad (15)$$

Then $(f_{\delta, L, \mu}(\bar{x}, X), g_{\delta, L, \mu}(\bar{x}, X))$ is a $(\delta, 2L_g, \mu_g/2)$ -model of f at point \bar{x} , i.e.

$$\frac{\mu_g}{4} \|\bar{y} - \bar{x}\|_2^2 \leq f(\bar{y}) - f_{\delta, L, \mu}(\bar{x}, X) - \langle g_{\delta, L, \mu}(\bar{x}, X), \bar{y} - \bar{x} \rangle \leq L_g \|\bar{y} - \bar{x}\|_2^2 + \delta.$$

The inexact oracle defined in Lemma 2 represents a $(\delta, 2L_g, \mu_g/2)$ -model of F . Note that it uses *global* strong convexity constants instead of *local* ones. Global constants may be significantly better for method performance, as pointed out in [134]. The lemma relates the projection accuracy δ' to inexact oracle parameter δ .

2.7.2 Convergence result for Algorithm 5

First, Lemma 2 states that $(\delta, 2L_g, \mu_g/2)$ -model of F is obtained if the gradient is computed in δ' -neighborhood of \mathbf{C} . In order to achieve this δ' -neighborhood, one needs to make a sufficient number of consensus (Algorithm 4) iterations.

Lemma 3. *Let consensus accuracy be maintained at level δ' , i.e. $\|U^j - \bar{U}^j\|_2^2 \leq \delta'$ for $j = 1, \dots, k$ and let Assumption 4 hold. Define*

$$\sqrt{D} := \left(\frac{2L_l}{\sqrt{L\mu}} + 1 \right) \sqrt{\delta'} + \frac{L_l}{\mu} \sqrt{n} \left(\|\bar{u}^0 - x^*\|_2^2 + \frac{8\delta'}{\sqrt{L\mu}} \right)^{1/2} + \frac{2\|\nabla F(X^*)\|_2}{\sqrt{L\mu}}.$$

Then it is sufficient to make $T_k = T = \frac{\tau}{2\lambda} \log \frac{D}{\delta'}$ consensus iterations (where τ and λ are defined in Assumption 4) in order to obtain consensus with δ' -accuracy on step $k+1$, i.e. $\|U^{k+1} - \bar{U}^{k+1}\|_2^2 \leq \delta'$.

A basis for the proof of Lemma 3 is a contraction property of mixing matrix sequence $\{\mathbf{M}^k\}_{k=0}^\infty$ (see Assumption 4).

Second, provided that projection accuracy on every step of Algorithm 5 is sustained at level δ' , the algorithm turns into an accelerated scheme with inexactness. Its convergence rate is given by the following

Lemma 4. *Provided that consensus accuracy is δ' , i.e. $\|U^j - \bar{U}^j\|_2^2 \leq \delta'$ for $j = 1, \dots, k$, we have*

$$\begin{aligned} f(\bar{x}^k) - f(x^*) &\leq \frac{\|\bar{u}^0 - x^*\|_2^2}{2A^k} + \frac{2\sum_{j=1}^k A^j \delta}{A^k} \\ \|\bar{u}^k - x^*\|_2^2 &\leq \frac{\|\bar{u}^0 - x^*\|_2^2}{1 + A^k \mu} + \frac{4\sum_{j=1}^k A^j \delta}{1 + A^k \mu} \end{aligned}$$

where δ is given in (15).

Finally, putting Lemmas 3 and 4 together yields a convergence result for Algorithm 5.

Theorem 5. *Recall the definitions of τ and λ from Assumption 2, choose some $\varepsilon > 0$ and set*

$$T_k = T = \frac{\tau}{2\lambda} \log \frac{D}{\delta'}, \quad \delta' = \frac{n\varepsilon}{32} \frac{\mu_g^{3/2}}{L_g^{1/2} L_l^2}.$$

Also define

$$\begin{aligned} D_1 &= \frac{L_l}{L_g^{1/2} \mu_g} \left[8\sqrt{2}L_l \|\bar{u}^0 - x^*\|_2 \left(\frac{L_g}{\mu_g} \right)^{3/4} + \frac{4\sqrt{2}\|\nabla F(X^*)\|_2}{\sqrt{n}} \left(\frac{L_g}{\mu_g} \right)^{1/4} \right], \\ D_2 &= \frac{L_l}{L_g^{1/2} \mu_g} \left[3\sqrt{\mu_g} + 4\sqrt{2n} \left(\frac{L_g}{\mu_g} \right)^{1/4} \right]. \end{aligned}$$

Then Algorithm 5 requires

$$N = 2\sqrt{\frac{L_g}{\mu_g}} \log \left(\frac{\|\bar{u}^0 - x^*\|_2^2}{2L_g \varepsilon} \right) \quad (17)$$

gradient computations at each node and

$$N_{tot} = N \cdot T = 2\sqrt{\frac{L_g}{\mu_g}} \frac{\tau}{\lambda} \cdot \log \left(\frac{2L_g \|\bar{u}^0 - x^*\|_2^2}{\varepsilon} \right) \log \left(\frac{D_1}{\sqrt{\varepsilon}} + D_2 \right) \quad (18)$$

communication steps to yield X^N such that

$$f(\bar{x}^N) - f(x^*) \leq \varepsilon, \quad \left\| X^N - \bar{X}^N \right\|_2^2 \leq \delta'.$$

In the time-static case, contraction term τ/λ turns into $\chi(\mathbf{M})$, and an accelerated consensus procedure of type (7) may be employed. This results in a better dependence on graph connectivity and leads to a complexity bound $O\left(\sqrt{\frac{L_g}{\mu_g}} \sqrt{\chi(\mathbf{M})} \log^2\left(\frac{1}{\varepsilon}\right)\right)$ which is optimal up to a logarithmic term. Similar results are attained in works which use penalty-based methods [93, 132, 51] (see Appendix B in [51]) for details.

Remark. The analysis of Algorithm 5 presented in [130] results in constants L_g, μ_g in the complexity bound. These constants are better than local constants L_i, μ_i , but still can be improved. The inexact oracle concept allows to reduce decentralized optimization problem to minimization of $f(x)$ over \mathbb{R}^d with inexact oracle. Therefore, the complexity will depend on constants L_f, μ_f which characterize f itself, not its flattened variant F . An accurate analysis on this issue is presented in Section 2.8.

2.7.3 Stochastic decentralized optimization

The technique used in Algorithm 5 can be extended to stochastic objectives. Following the definitions in [130], let $f_i(x) := \mathbb{E}_{\xi_i} f_i(x, \xi_i)$, where ξ_i 's are random variables. Variables ξ_i represent the source of stochasticity in $f_i(x, \xi_i)$ which may be caused by random sampling or stochastic noise. For each $i = 1, \dots, n$ we assume that $\nabla f_i(x, \xi_i)$ is $L_i(\xi)$ continuous and there exists a constant $L_i \geq 0$ such that $\sqrt{\mathbb{E}_{\xi_i} L_i(\xi_i)^2} \leq L_i < +\infty$. Under these assumptions f_i is L_i -smooth. We also bound the variance of $\nabla f_i(x, \xi_i)$:

$$\mathbb{E}_{\xi_i} [\|\nabla f_i(x, \xi_i) - \nabla f_i(x)\|_2^2] \leq \sigma_i^2.$$

Let us define $\sigma_g^2 = \frac{1}{m} \sum_{i=1}^m \sigma_i^2$. The algorithm in [130] combines a consensus subroutine technique similar to Algorithm 5 and also uses a specific batch-size policy. In order to analyze the method, inexact oracle framework similar to that of Section 2.7 is used. The inexactness of gradient has two sources: inexact projection onto the constraint set via consensus subroutine and stochastic noise. On the one hand, tuning the batch size allows to reduce the variance of the batched gradient at the cost of additional stochastic oracle calls. Therefore, proper batch size guarantees a balance between the stochastic gradient noise and the number of gradient calculations. On the other hand, the accuracy of the consensus is tuned by the choice of number of consensus iterations. Choosing a proper batch size and number of consensus iterations allows to obtain optimal complexities both in the number of computations and communications up to a logarithmic factor. Namely, the method in [130] requires $\tilde{O}\left(\max\left\{\frac{\sigma_g^2}{n\mu_g\varepsilon}, \sqrt{\frac{L_g}{\mu_g}} \log \frac{1}{\varepsilon}\right\}\right)$ oracle calls per node. In the time-varying case, it requires $\tilde{O}\left(\frac{\tau}{\lambda} \sqrt{\frac{L_g}{\mu_g}}\right)$ communication rounds (where τ and λ are defined in Assumption 4), and in time-static case its communication complexity takes the form $\tilde{O}\left(\sqrt{\frac{L_g}{\mu_g}} \chi\right)$ and is achieved by using Chebyshev acceleration.

2.8 Decentralized Saddle-Point Problems

Along with minimization problems, sum-type min-max problems of type

$$\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} f(x, y) := \frac{1}{m} \sum_{i=1}^m f_i(x, y)$$

where \mathcal{X} and \mathcal{Y} are convex compacts, can be solved in a decentralized manner, as well. The same way as in Assumptions 2, 3 for minimization tasks, we introduce assumptions for min-max problems.

Assumption 6 For every $i = 1, \dots, m$, function f_i is differentiable, convex in x , concave in y and L_i -smooth.

Assumption 7 Function f is μ -strongly-convex in x , μ_f -strongly-concave in y ($\mu_f > 0$) and L_f -smooth.

Saddle-point problems have many practical applications: classical and well-studied in economy and in game theory [161, 40], and modern in imaging denoising [24], in adversarial training [9, 50], and in statistical learning [1]. But dis-

tributed saddle-point problems is not as widely studied in the literature as the minimization problems. Let us highlighted the main works devoted to decentralized min-max problems. Most of the works are devoted to decentralized algorithms on fixed graph topology. In paper [20], the authors present lower bounds for deterministic decentralized saddle point problems under Assumptions 6 and 7. These estimates are as follows

$$\begin{aligned} \Omega\left(\frac{L_f}{\mu_f} \log(1/\varepsilon)\right) & \quad \text{computation complexity and} \\ \Omega\left(\sqrt{\chi} \frac{L_f}{\mu_f} \log(1/\varepsilon)\right) & \quad \text{communication complexity.} \end{aligned} \quad (19)$$

Additionally, the paper provides an optimal algorithm (up to logarithmic factors), which achieves the lower bounds. Among the disadvantages of the Algorithm presented in [20], one can single out multiple gossip steps, this approach is unstable and not a popular in practice. A similar Algorithm with multiple gossip steps is proposed in [102], but they consider convergence in the non-convex case (under the minty condition [107, 66]). Also, this work shows the effectiveness of decentralized training of GANs. [104] is also devoted to minty non-convex saddle-point problems. It is also interesting to note the work [20] on saddle-point problems in terms of data-similarity. The work gives lower bounds for communication complexity, as well as optimal algorithms for such setting of the problem. In particular, the lower and upper bounds state that

$$\sqrt{\chi} \left(1 + \frac{\delta}{\mu_f}\right) \log(1/\varepsilon) \quad \text{communication rounds}$$

are enough to achieve ε -precision. Interesting to note, that for uniformly distributed data, with high probability $L_i \approx L_f$ and $\delta \sim \tilde{O}(\max_i L_i / \sqrt{n})$, where n – the number of local samples on each node. This means that the data-similarity bounds on communication rounds is significantly better than the general one (19).

It is also important to mention the works devoted to saddle-point problems on time-varying networks. In particular, paper [18] is devoted to lower bounds and optimal algorithms for connected topology. Work [14] is devoted to the broader case of time-varying networks, for example, methods can do local steps (iterations without communication).

An interesting variant of saddle-point problems are extensions to local and global variables, i.e. problems of the form

$$\min_{p, \{x_i\}_{i=1}^m} \max_{r, \{y_i\}_{i=1}^m} \frac{1}{m} \sum_{i=1}^m f_i(x_i, p, y_i, r). \quad (20)$$

Applications of problems of this type minimization tasks with separable and semi-definite constraints [106], decentralized reinforcement learning [162] and distributed computation of Wasserstein barycenters [129, 35]. A subgradient method for problems of type (20) was proposed in [106]. The method has a $O(1/\sqrt{N})$ convergence rate. A recent work [129] proposed a method based on Mirror-Prox, capable of working in general proximal setup, reaching a $O(1/N)$ convergence rate and an accelerated rate on χ . The method achieves optimal oracle and communication complexities in Euclidean convex-concave case over time-static graphs.

3 Convex Problems with Affine Constraints

In this section¹, we consider convex optimization problem with affine constraints

$$\min_{Ax=0, x \in Q} f(x), \quad (21)$$

where $A \succeq 0$, $\text{Ker}A \neq \{0\}$ and Q is a closed convex subset of \mathbb{R}^n . Up to a sign, the dual problem is defined as follows:

$$\min_y \psi(y), \quad \text{where} \quad (22)$$

$$\varphi(y) = \max_{x \in Q} \{\langle y, x \rangle - f(x)\}, \quad (23)$$

$$\psi(y) = \varphi(A^\top y) = \max_{x \in Q} \{\langle y, Ax \rangle - f(x)\} = \langle A^\top y, x(A^\top y) \rangle - f(x(A^\top y)), \quad (24)$$

¹ The narrative in this section follows [51].

where $x(y) \stackrel{\text{def}}{=} \operatorname{argmax}_{x \in Q} \{ \langle y, x \rangle - f(x) \}$. Since $\operatorname{Ker} A \neq \{0\}$ the solution of the dual problem (22) is not unique. We use y^* to denote the solution of (22) with the smallest ℓ_2 -norm $R_y \stackrel{\text{def}}{=} \|y^*\|_2$.

3.1 Primal Approach

In this section, we focus on primal approaches to solve (21) and, in particular, the main goal of this section is to present first-order methods that are optimal both in terms of $\nabla f(x)$ and $A^\top Ax$ calculations. One can apply the following trick [32, 44, 51] to solve problem (21): instead of (21) one can solve penalized problem

$$\min_{x \in Q} \left\{ F(x) = f(x) + \frac{R_y^2}{\varepsilon} \|Ax\|_2^2 \right\}, \quad (25)$$

where $\varepsilon > 0$ is the desired accuracy of the solution in terms of $f(x)$ that we want to achieve (see the details in [51]).

Next, we assume that f is μ -strongly convex, but possibly non-smooth function with bounded (sub) gradients: $\|\nabla f(x)\|_2 \leq M$ for all $x \in Q$. In this setting, one can apply Sliding algorithm from [85, 84] to get optimal rates of convergence. The method is presented as Algorithm 6 and it is aimed to solve the following problem:

$$\min_{x \in Q} \{ \Psi(x) = h(x) + f(x) \}, \quad (26)$$

where $h(x)$ is convex and L -smooth, $f(x)$ is convex, but can be non-smooth, and x^* is an arbitrary solution of the problem. In this case, it is additionally assumed that $f(x)$ has uniformly bounded subgradients: there exists non-negative constant M such that² $\|\nabla f(x)\|_2 \leq M$ for all $x \in Q$ and all subgradients at this point $\nabla f(x) \in \partial f(x)$.

Algorithm 6 Sliding Algorithm [85, 84]

Input: Initial point $x_0 \in Q$ and iteration limit N .

Let $\beta_k \in \mathcal{R}_{++}$, $\gamma_k \in \mathcal{R}_+$, and $T_k \in \mathbb{N}$, $k = 1, 2, \dots$, be given and set $\bar{x}_0 = x_0$.

for $k = 1, 2, \dots, N$ **do**

1. Set $\underline{x}_k = (1 - \gamma_k)\bar{x}_{k-1} + \gamma_k x_{k-1}$, and let $h_k(\cdot) \equiv l_h(\underline{x}_k, \cdot)$, where $l_h(x, y) = h(x) + \langle \nabla h(x), y - x \rangle$.

2. Set

$$(x_k, \bar{x}_k) = \text{PS}(h_k, x_{k-1}, \beta_k, T_k).$$

3. Set $\bar{x}_k = (1 - \gamma_k)\bar{x}_{k-1} + \gamma_k \bar{x}_k$.

end for

Output: \bar{x}_N .

The PS (prox-sliding) procedure.

procedure: $(x^+, \bar{x}^+) = \text{PS}(g, x, \beta, T)$

Let the parameters $p_t \in \mathbb{R}_{++}$ and $\theta_t \in [0, 1]$, $t = 1, \dots$, be given. Set $u_0 = \tilde{u}_0 = x$.

for $t = 1, 2, \dots, T$ **do**

$$u_t = \operatorname{argmin}_{u \in Q} \left\{ g(u) + l_f(u_{t-1}, u) + \frac{\beta}{2} \|u - x\|_2^2 + \frac{\beta p_t}{2} \|u - u_{t-1}\|_2^2 \right\},$$

$$\tilde{u}_t = (1 - \theta_t)\tilde{u}_{t-1} + \theta_t u_t,$$

where $l_f(x, y) = f(x) + \langle \nabla f(x), y - x \rangle$.

end for

Set $x^+ = u_T$ and $\bar{x}^+ = \tilde{u}_T$.

end procedure:

The key property of Algorithm 6 is its ability to separate oracle complexities for smooth and non-smooth parts of the objective. That is, to find such \hat{x} that $\Psi(\hat{x}) - \Psi(x^*) \leq \varepsilon$ Sliding requires

$$O\left(\sqrt{\frac{LR^2}{\varepsilon}}\right) \text{ calculations of } \nabla h(x)$$

² For the sake of simplicity, we slightly abuse the notation and denote gradients and subgradients similarly.

and

$$O\left(\frac{M^2 R^2}{\varepsilon^2} + \sqrt{\frac{LR^2}{\varepsilon}}\right) \text{ calculations of } \nabla f(x),$$

where $R = \|x^0 - x^*\|_2$.

Now, we go back to the problem (25) and consider the case when $\mu = 0$. In these settings, to find \hat{x} such that

$$F(\hat{x}) - F(x^*) \leq \varepsilon \quad (27)$$

one can run Algorithm 6 considering $f(x)$ as the non-smooth term and $R_y^2/\varepsilon\|Ax\|_2^2$ as the smooth one. In this case, Sliding requires

$$O\left(\sqrt{\frac{\lambda_{\max}(A^\top A)R_y^2 R^2}{\varepsilon^2}}\right) \text{ calculations of } A^\top Ax, \quad (28)$$

$$O\left(\frac{M^2 R^2}{\varepsilon^2}\right) \text{ calculations of } \nabla f(x). \quad (29)$$

Next, we consider the situation when Q is a compact set, $\nabla f(x)$ is not available, and unbiased stochastic gradient $\nabla f(x, \xi)$ is used instead:

$$\|\mathbb{E}_\xi [\nabla f(x, \xi)] - \nabla f(x)\|_2 \leq \delta, \quad (30)$$

$$\mathbb{E}_\xi \left[\exp\left(\frac{\|\nabla f(x, \xi) - \mathbb{E}_\xi [\nabla f(x, \xi)]\|_2^2}{\sigma^2}\right) \right] \leq \exp(1), \quad (31)$$

where $\delta \geq 0$ and $\sigma \geq 0$. When $\delta = 0$, i.e., stochastic gradients are unbiased, one can show [85, 84] that Stochastic Sliding (S-Sliding) method can achieve (27) with probability at least $1 - \beta$, $\beta \in (0, 1)$ requiring the same number of calculations of $A^\top Ax$ as in (28) up to logarithmic factors and

$$\tilde{O}\left(\frac{(M^2 + \sigma^2)R^2}{\varepsilon^2}\right) \text{ calculations of } \nabla f(x, \xi). \quad (32)$$

When $\mu > 0$ one can apply restarts technique for S-Sliding and get the method (RS-Sliding) [32, 158] that guarantees (27) with probability at least $1 - \beta$, $\beta \in (0, 1)$ using

$$\tilde{O}\left(\sqrt{\frac{\lambda_{\max}(A^\top A)R_y^2}{\mu\varepsilon}}\right) \text{ calculations of } A^\top Ax, \quad (33)$$

$$\tilde{O}\left(\frac{M^2 + \sigma^2}{\mu\varepsilon}\right) \text{ calculations of } \nabla f(x, \xi). \quad (34)$$

We notice that bounds presented above for the non-smooth case are proved when Q is bounded. For the case of unbounded Q the convergence results with such rates were established only in expectation. Moreover, it would be interesting to study S-Sliding and RS-Sliding in the case when $\delta > 0$, i.e., stochastic gradient is biased.

3.2 Dual Approach

In this section, we assume that one can construct a dual problem for (21). If f is μ -strongly convex in ℓ_2 -norm, then ψ and φ have L_ψ -Lipschitz continuous and L_φ -Lipschitz continuous in ℓ_2 -norm gradients respectively [69, 128], where $L_\psi = \lambda_{\max}(A^\top A)/\mu$ and $L_\varphi = 1/\mu$. In our proofs, we often use Demyanov–Danskin theorem [128] which states that

$$\nabla \psi(y) = Ax(A^\top y), \quad \nabla \varphi(y) = x(y). \quad (35)$$

Moreover, we do not assume that A is symmetric or positive semidefinite.

Below we propose a primal-dual method for the case when f is additionally Lipschitz continuous on some ball and two methods for the problems when the primal function is also L -smooth and Lipschitz continuous on some ball. In the subsections below, we assume that $Q = \mathbb{R}^n$. The formal proofs of the presented results are given in [51].

3.2.1 Convex Dual Function

In this section, we assume that the dual function $\varphi(y)$ could be rewritten as an expectation, i.e., $\varphi(y) = \mathbb{E}_\xi [\varphi(y, \xi)]$, where stochastic realizations $\varphi(y, \xi)$ are differentiable in y functions almost surely in ξ . Then, we can also represent $\psi(y)$ as an expectation: $\psi(y) = \mathbb{E}_\xi [\psi(y, \xi)]$. Consider the stochastic function $f(x, \xi)$ which is defined implicitly as follows:

$$\varphi(y, \xi) = \max_{x \in \mathbb{R}^n} \{ \langle y, x \rangle - f(x, \xi) \}. \quad (36)$$

Similarly to the deterministic case, we introduce $x(y, \xi) \stackrel{\text{def}}{=} \operatorname{argmax}_{x \in \mathbb{R}^n} \{ \langle y, x \rangle - f(x, \xi) \}$ which satisfies $\nabla \varphi(y, \xi) = x(y, \xi)$ due to Demyanov–Danskın theorem, where the gradient is taken w.r.t. y . As a simple corollary, we get $\nabla \psi(y, \xi) = Ax(A^\top y)$. Finally, introduced notations and obtained relations imply that $x(y) = \mathbb{E}_\xi [x(y, \xi)]$ and $\nabla \psi(y) = \mathbb{E}_\xi [\nabla \psi(y, \xi)]$.

Consider the situation when $x(y, \xi)$ is known only through the noisy observations $\tilde{x}(y, \xi) = x(y, \xi) + \delta(y, \xi)$ and assume that the noise is bounded in expectation, i.e., there exists non-negative deterministic constant $\delta_y \geq 0$, such that

$$\|\mathbb{E}_\xi [\delta(y, \xi)]\|_2 \leq \delta_y, \quad \forall y \in \mathbb{R}^n. \quad (37)$$

Assume additionally that $\tilde{x}(y, \xi)$ satisfies so-called ‘‘light-tails’’ inequality:

$$\mathbb{E}_\xi \left[\exp \left(\frac{\|\tilde{x}(y, \xi) - \mathbb{E}_\xi [\tilde{x}(y, \xi)]\|_2^2}{\sigma_x^2} \right) \right] \leq \exp(1), \quad \forall y \in \mathbb{R}^n, \quad (38)$$

where σ_x is some positive constant. It implies that we have an access to the biased stochastic gradient $\tilde{\nabla} \psi(y, \xi) \stackrel{\text{def}}{=} A\tilde{x}(y, \xi)$ which satisfies following relations:

$$\|\mathbb{E}_\xi [\tilde{\nabla} \psi(y, \xi)] - \nabla \psi(y)\|_2 \leq \delta, \quad \forall y \in \mathbb{R}^n, \quad (39)$$

$$\mathbb{E}_\xi \left[\exp \left(\frac{\|\tilde{\nabla} \psi(y, \xi) - \mathbb{E}_\xi [\tilde{\nabla} \psi(y, \xi)]\|_2^2}{\sigma_\psi^2} \right) \right] \leq \exp(1), \quad \forall y \in \mathbb{R}^d, \quad (40)$$

where $\delta \stackrel{\text{def}}{=} \sqrt{\lambda_{\max}(A^\top A)} \delta_y$ and $\sigma_\psi \stackrel{\text{def}}{=} \sqrt{\lambda_{\max}(A^\top A)} \sigma_x$. We will use $\tilde{\nabla} \Psi(y, \xi^k)$ to denote batched stochastic gradient:

$$\tilde{\nabla} \Psi(y, \xi^k) = \frac{1}{r_k} \sum_{l=1}^{r_k} \tilde{\nabla} \psi(y, \xi^l), \quad \tilde{x}(y, \xi^k) = \frac{1}{r_k} \sum_{l=1}^{r_k} \tilde{x}(y, \xi^l) \quad (41)$$

The size of the batch r_k could always be restored from the context, so, we do not specify it here. Note that the batch version satisfies (see the details in [51])

$$\|\mathbb{E} [\tilde{\nabla} \Psi(x, \xi^k)] - \nabla \psi(x)\|_2 \leq \delta, \quad \forall x \in \mathbb{R}^n, \quad (42)$$

$$\mathbb{E} \left[\exp \left(\frac{\|\tilde{\nabla} \Psi(x, \xi^k) - \mathbb{E} [\tilde{\nabla} \Psi(x, \xi^k)]\|_2^2}{O(\sigma_\psi^2/r_k^2)} \right) \right] \leq \exp(1), \quad \forall x \in \mathbb{R}^n, \quad (43)$$

In these settings, we consider a method called SPDSTM (Stochastic Primal-Dual Similar Triangles Method, see Algorithm 7). Note that Algorithm 4 from [34] is a special case of SPDSTM when $\delta = 0$, i.e., stochastic gradient is unbiased, up to a factor 2 in the choice of \tilde{L} .

Below we present the main convergence result of this section.

Theorem 8 (Theorem 5.1 from [51]). *Assume that f is μ -strongly convex and $\|\nabla f(x^*)\|_2 = M_f$. Let $\varepsilon > 0$ be a desired accuracy. Next, assume that f is L_f -Lipschitz continuous on the ball $B_{R_f}(0)$ with*

Algorithm 7 SPDSTM**Input:** $\tilde{y}^0 = z^0 = y^0 = 0$, number of iterations N , $\alpha_0 = A_0 = 0$ 1: **for** $k = 0, \dots, N$ **do**2: Set $\tilde{L} = 2L\psi$ 3: Set $A_{k+1} = A_k + \alpha_{k+1}$, where $2\tilde{L}\alpha_{k+1}^2 = A_k + \alpha_{k+1}$ 4: $\tilde{y}^{k+1} = (A_k y^k + \alpha_{k+1} z^k) / A_{k+1}$ 5: $z^{k+1} = z^k - \alpha_{k+1} \tilde{\nabla} \Psi(\tilde{y}^{k+1}, \xi^k)$ 6: $y^{k+1} = (A_k y^k + \alpha_{k+1} z^{k+1}) / A_{k+1}$ 7: **end for****Output:** $y^N, \tilde{x}^N = \frac{1}{A_N} \sum_{k=0}^N \alpha_k \tilde{x}(A^\top \tilde{y}^k, \xi^k)$.

$$R_f = \tilde{\Omega} \left(\max \left\{ \frac{R_y}{A_N \sqrt{\lambda_{\max}(A^\top A)}}, \frac{\sqrt{\lambda_{\max}(A^\top A)} R_y}{\mu}, R_x \right\} \right),$$

where R_y is such that $\|y^*\|_2 \leq R_y$, y^* is the solution of the dual problem (22), and $R_x = \|x(A^\top y^*)\|_2$. Assume that at iteration k of Algorithm 7 batch size is chosen according to the formula $r_k \geq \max \left\{ 1, \frac{\sigma_\psi^2 \tilde{\alpha}_k \ln(N/\beta)}{\hat{C}\varepsilon} \right\}$, where $\tilde{\alpha}_k = \frac{k+1}{2L}$, $0 < \varepsilon \leq \frac{H\tilde{L}R_0^2}{N^2}$, $0 \leq \delta \leq \frac{G\tilde{L}R_0}{(N+1)^2}$ and $N \geq 1$ for some numeric constant $H > 0$, $G > 0$ and $\hat{C} > 0$. Then with probability $\geq 1 - 4\beta$, where $\beta \in (0, 1/8)$, after $N = \tilde{O} \left(\sqrt{\frac{M_f}{\mu\varepsilon} \chi(A^\top A)} \right)$ iterations where $\chi(A^\top A) = \frac{\lambda_{\max}(A^\top A)}{\lambda_{\min}^+(A^\top A)}$, the outputs \tilde{x}^N and y^N of Algorithm 7 satisfy the following condition

$$f(\tilde{x}^N) - f(x^*) \leq f(\tilde{x}^N) + \psi(y^N) \leq \varepsilon, \quad \|A\tilde{x}^N\|_2 \leq \frac{\varepsilon}{R_y} \quad (44)$$

with probability at least $1 - 4\beta$. What is more, to guarantee (44) with probability at least $1 - 4\beta$ Algorithm 7 requires

$$\tilde{O} \left(\max \left\{ \frac{\sigma_x^2 M_f^2}{\varepsilon^2} \chi(A^\top A) \ln \left(\frac{1}{\beta} \sqrt{\frac{M_f}{\mu\varepsilon} \chi(A^\top A)} \right), \sqrt{\frac{M_f}{\mu\varepsilon} \chi(A^\top A)} \right\} \right) \quad (45)$$

calls of the biased stochastic oracle $\tilde{\nabla} \psi(y, \xi)$, i.e. $\tilde{x}(y, \xi)$.

3.2.2 Strongly Convex Dual Functions and Restarts Technique

In this section, we assume that primal functional f is additionally L -smooth. It implies that the dual function ψ in (22) is additionally μ_ψ -strongly convex in $y^0 + (\text{Ker} A^\top)^\perp$ where $\mu_\psi = \lambda_{\min}^+(A^\top A)/L$ [69, 128] and $\lambda_{\min}^+(A^\top A)$ is the minimal positive eigenvalue of $A^\top A$.

From weak duality $-f(x^*) \leq \psi(y^*)$ and (24) we get the key relation of this section (see also [6, 7, 116])

$$f(x(A^\top y)) - f(x^*) \leq \langle \nabla \psi(y), y \rangle = \langle Ax(A^\top y), y \rangle. \quad (46)$$

This inequality implies the following theorem.

Theorem 9 (Theorem 5.2 from [51]). Consider function f and its dual function ψ defined in (24) such that problems (21) and (22) have solutions. Assume that y^N is such that $\|\nabla \psi(y^N)\|_2 \leq \varepsilon/R_y$ and $y^N \leq 2R_y$, where $\varepsilon > 0$ is some positive number and $R_y = \|y^*\|_2$ where y^* is any minimizer of ψ . Then for $x^N = x(A^\top y^N)$ following relations hold:

$$f(x^N) - f(x^*) \leq 2\varepsilon, \quad \|Ax^N\|_2 \leq \frac{\varepsilon}{R_y}, \quad (47)$$

where x^* is any minimizer of f .

That is why, in this section we mainly focus on the methods that provide optimal convergence rates for the gradient norm. In particular, we consider Recursive Regularization Meta-Algorithm from (see Algorithm 8) [43] with AC-SA² (see Algorithm 10) as a subroutine (i.e. RRMA-AC-SA²) which is based on AC-SA algorithm (see Algorithm 9) from [48]. We notice that RRMA-AC-SA² is applied for a regularized dual function

$$\tilde{\psi}(y) = \psi(y) + \frac{\lambda}{2} \|y - y^0\|_2^2, \quad (48)$$

where $\lambda > 0$ is some positive number which will be defined further. Function $\tilde{\psi}$ is λ -strongly convex and \tilde{L}_ψ -smooth in \mathbb{R}^n where $\tilde{L}_\psi = L_\psi + \lambda$. For now, we just assume w.l.o.g. that $\tilde{\psi}$ is $(\mu_\psi + \lambda)$ -strongly convex in \mathbb{R}^n , but we will go back to this question further.

In this section we consider the same oracle as in Section 3.2.1, but we additionally assume that $\delta = 0$, i.e., stochastic first-order oracle is unbiased. To define batched version of the stochastic gradient we will use the following notation:

$$\nabla\Psi(y, \xi^t, r_t) = \frac{1}{r_t} \sum_{l=1}^{r_t} \nabla\psi(y, \xi^l), \quad x(y, \xi^t, r_t) = \frac{1}{r_t} \sum_{l=1}^{r_t} x(y, \xi^l). \quad (49)$$

As before, in the cases when the batch-size r_t can be restored from the context, we will use simplified notation $\nabla\Psi(y, \xi^t)$ and $x(y, \xi^t)$.

Algorithm 8 RRMA-AC-SA² [43]

Input: y^0 — starting point, m — total number of iterations

- 1: $\psi_0 \leftarrow \tilde{\psi}, \hat{y}^0 \leftarrow y^0, T \leftarrow \lceil \log_2 \frac{\tilde{L}_\psi}{\lambda} \rceil$
 - 2: **for** $k = 1, \dots, T$ **do**
 - 3: Run AC-SA² for m/T iterations to optimize ψ_{k-1} with \hat{y}^{k-1} as a starting point and get the output \hat{y}^k
 - 4: $\psi_k(y) \leftarrow \tilde{\psi}(y) + \lambda \sum_{l=1}^k 2^{l-1} \|y - \hat{y}^l\|_2^2$
 - 5: **end for**
- Output:** \hat{y}^T .
-

In the AC-SA algorithm we use batched stochastic gradients of functions ψ_k which are defined as follows:

$$\begin{aligned} \nabla\Psi_k(y, \xi^t) &= \frac{1}{r_t} \sum_{l=1}^{r_t} \nabla\psi_k(y, \xi^l), \\ \nabla\psi_k(y, \xi) &= \nabla\psi(y, \xi) + \lambda(y - y^0) + \lambda \sum_{l=1}^k 2^l (y - \hat{y}^l). \end{aligned} \quad (50)$$

Algorithm 9 AC-SA [48]

Input: z^0 — starting point, m — number of iterations, ψ_k — objective function

- 1: $y_{ag}^0 \leftarrow z^0, y_{md}^0 \leftarrow z^0$
 - 2: **for** $t = 1, \dots, m$ **do**
 - 3: $\alpha_t \leftarrow \frac{2}{t+1}, \gamma_t \leftarrow \frac{4\tilde{L}_\psi}{t(t+1)}$
 - 4: $y_{md}^t \leftarrow \frac{(1-\alpha_t)(\lambda+\gamma_t)}{\gamma_t+(1-\alpha_t^2)\lambda} y_{ag}^{t-1} + \frac{\alpha_t((1-\alpha_t)\lambda+\gamma_t)}{\gamma_t+(1-\alpha_t^2)\lambda} z^{t-1}$
 - 5: $z^t \leftarrow \frac{\alpha_t \lambda}{\lambda+\gamma_t} y_{md}^t + \frac{(1-\alpha_t)\lambda+\gamma_t}{\lambda+\gamma_t} z^{t-1} - \frac{\alpha_t}{\lambda+\gamma_t} \nabla\Psi_k(y_{md}^t, \xi^t)$
 - 6: $y_{ag}^t \leftarrow \alpha_t z^t + (1-\alpha_t) y_{ag}^{t-1}$
 - 7: **end for**
- Output:** y_{ag}^m .
-

Algorithm 10 AC-SA² [43]

Input: z^0 — starting point, m — number of iterations, ψ_k — objective function

- 1: Run AC-SA for $m/2$ iterations to optimize ψ_k with z^0 as a starting point and get the output y^1
 - 2: Run AC-SA for $m/2$ iterations to optimize ψ_k with y^1 as a starting point and get the output y^2
- Output:** y^2 .
-

The following theorem states the main result for RRMA-AC-SA² that we need in the section.

Theorem 10 (Corollary 1 from [43]). *Let ψ be L_ψ -smooth and μ_ψ -strongly convex function and $\lambda = \Theta((L_\psi \ln^2 N)/N^2)$ for some $N > 1$. If the Algorithm 8 performs N iterations in total³ with batch size r for all iterations, then it will provide such a point \hat{y} that*

$$\mathbb{E} [\|\nabla\psi(\hat{y})\|_2^2 | y^0, r] \leq C \left(\frac{L_\psi^2 \|y^0 - y^*\|_2^2 \ln^4 N}{N^4} + \frac{\sigma_\psi^2 \ln^6 N}{rN} \right), \quad (51)$$

where $C > 0$ is some positive constant and y^* is a solution of the dual problem (22).

The following result shows that w.l.o.g. we can assume that function ψ defined in (24) is μ_ψ -strongly convex everywhere with $\mu_\psi = \lambda_{\min}^+(A^\top A)/L$. In fact, from L -smoothness of f we have only that ψ is μ_ψ -strongly convex in $y^0 + (\text{Ker}(A^\top))^\perp$ (see [69, 128] for the details). However, the structure of the considered here methods is such that all points generated by the RRMA-AC-SA² and, in particular, AC-SA lie in $y^0 + (\text{Ker}(A^\top))^\perp$.

Theorem 11 (Theorem 5.4 from [51]). *Assume that Algorithm 9 is run for the objective $\psi_k(y) = \tilde{\psi}(y) + \lambda \sum_{l=1}^k 2^{l-1} \|y - \hat{y}^l\|_2^2$ with z^0 as a starting point, where $z^0, \hat{y}^1, \dots, \hat{y}^k$ are some points from $y^0 + (\text{Ker}(A^\top))^\perp$ and $y^0 \in \mathbb{R}^n$. Then for all $t \geq 0$ we have $y_{nd}^t, z^t, y_{ag}^t \in y^0 + (\text{Ker}(A^\top))^\perp$.*

Corollary 1 (Corollary 5.5 from [51]). *Assume that Algorithm 8 is run for the objective $\psi_k(y) = \tilde{\psi}(y) + \lambda \sum_{l=1}^k 2^{l-1} \|y - \hat{y}^l\|_2^2$ with y^0 as a starting point. Then for all $k \geq 0$ we have $\hat{y}^k \in y^0 + (\text{Ker}(A^\top))^\perp$.*

Now we are ready to present our approach⁴ of constructing an accelerated method for the strongly convex dual problem using restarts of RRMA-AC-SA². To explain the main idea we start with the simplest case: $\sigma_\psi^2 = 0$, $r = 0$. It means that there is no stochasticity in the method and the bound (51) can be rewritten in the following form:

$$\|\nabla\psi(\hat{y})\|_2 \leq \frac{\sqrt{C} L_\psi \|y^0 - y^*\|_2 \ln^2 N}{N^2} \leq \frac{\sqrt{C} L_\psi \|\nabla\psi(y^0)\|_2 \ln^2 N}{\mu_\psi N^2}, \quad (52)$$

where we used inequality $\|\nabla\psi(y^0)\|_2 \geq \mu_\psi \|y^0 - y^*\|$ which follows from the μ_ψ -strong convexity of ψ . It implies that after $\bar{N} = \tilde{O}(\sqrt{L_\psi/\mu_\psi})$ iterations of RRMA-AC-SA² the method returns such $\bar{y}^1 = \hat{y}$ that $\|\nabla\psi(\bar{y}^1)\|_2 \leq \frac{1}{2} \|\nabla\psi(y^0)\|_2$. Next, applying RRMA-AC-SA² with \bar{y}^1 as a starting point for the same number of iterations we will get new point \bar{y}^2 such that $\|\nabla\psi(\bar{y}^2)\|_2 \leq \frac{1}{2} \|\nabla\psi(\bar{y}^1)\|_2 \leq \frac{1}{4} \|\nabla\psi(y^0)\|_2$. Then, after $l = O(\ln(R_y \|\nabla\psi(y^0)\|_2/\epsilon))$ of such restarts we can get the point \bar{y}^l such that $\|\nabla\psi(\bar{y}^l)\|_2 \leq \epsilon/R_y$ with total number of gradients computations $\bar{N}l = \tilde{O}(\sqrt{L_\psi/\mu_\psi} \ln(R_y \|\nabla\psi(y^0)\|_2/\epsilon))$.

When $\sigma_\psi^2 \neq 0$ we need to modify this approach. The first ingredient to handle the stochasticity is large enough batch size for the l -th restart: r_l should be $\Omega(\sigma_\psi^2/(\bar{N} \|\nabla\psi(\bar{y}^{l-1})\|_2^2))$. However, in the stochastic case we do not have an access to the $\nabla\psi(\bar{y}^{l-1})$, so, such batch size is impractical. One possible way to fix this issue is to independently sample large enough number $\hat{r}_l \sim R_y^2/\epsilon^2$ of stochastic gradients additionally, which is the second ingredient of our approach, in order to get good enough approximation $\nabla\Psi(\bar{y}^{l-1}, \xi^{l-1}, \hat{r}_l)$ of $\nabla\psi(\bar{y}^{l-1})$ and use the norm of such an approximation which is close to the norm of the true gradient with big enough probability in order to estimate needed batch size r^l for the optimization procedure. Using this, we can get the bound of the following form:

$$\mathbb{E} \left[\|\nabla\psi(\bar{y}^l)\|_2^2 | \bar{y}^{l-1}, r_l, \hat{r}_l \right] \leq A_l \stackrel{\text{def}}{=} \frac{\|\nabla\psi(\bar{y}^{l-1})\|_2^2}{8} + \frac{\|\nabla\Psi(\bar{y}^{l-1}, \xi^{l-1}, \hat{r}_l) - \nabla\psi(\bar{y}^{l-1})\|_2^2}{32}.$$

The third ingredient is the amplification trick: we run $p_l = \Omega(\ln(1/\beta))$ independent trajectories of RRMA-AC-SA², get points $\bar{y}^{l,1}, \dots, \bar{y}^{l,p_l}$ and choose such $\bar{y}^{l,p(l)}$ among of them that $\|\nabla\psi(\bar{y}^{l,p(l)})\|_2$ is close enough to $\min_{p=1, \dots, p_l} \|\nabla\psi(\bar{y}^{l,p})\|_2$ with high probability, i.e., $\|\nabla\psi(\bar{y}^{l,p(l)})\|_2^2 \leq 2 \min_{p=1, \dots, p_l} \|\nabla\psi(\bar{y}^{l,p})\|_2^2 + \epsilon^2/8R_y^2$ with probability at least $1 - \beta$ for fixed $\nabla\Psi(\bar{y}^{l-1}, \xi^{l-1}, \hat{r}_l)$. We achieve it due to additional sampling of $\bar{r}_l \sim R_y^2/\epsilon^2$ stochastic gradients at $\bar{y}^{l,p}$ for each trajectory and choosing such $p(l)$ corresponding to the smallest norm of the obtained batched stochastic gradient. By Markov's inequality for all $p = 1, \dots, p_l$

$$\mathbb{P} \left\{ \|\nabla\psi(\bar{y}^{l,p})\|_2^2 \geq 2A_l | \bar{y}^{l-1}, r_l, \hat{r}_l \right\} \leq \frac{1}{2},$$

hence

³ The overall number of performed iterations during the calls of AC-SA² equals N .

⁴ This approach was described in [32] and formally proved in [51].

$$\mathbb{P} \left\{ \min_{p=1, \dots, p_l} \|\nabla \Psi(\bar{y}^{l,p})\|_2^2 \geq 2A_l | \bar{y}^{l-1}, r_l, \bar{r}_l \right\} \leq \frac{1}{2^{p_l}}.$$

That is, for $p_l = \log_2(1/\beta)$ we have that with probability at least $1 - 2\beta$

$$\|\nabla \Psi(\bar{y}^{l,p(l)})\|_2^2 \leq \frac{\|\nabla \Psi(\bar{y}^{l-1})\|_2^2}{2} + \frac{\|\nabla \Psi(\bar{y}^{l-1}, \xi^{l-1}, \hat{r}_l) - \nabla \Psi(\bar{y}^{l-1})\|_2^2}{8} + \frac{\varepsilon^2}{8R_y^2}$$

for fixed $\nabla \Psi(\bar{y}^{l-1}, \xi^{l-1}, \hat{r}_l)$ which means that

$$\|\nabla \Psi(\bar{y}^{l,p(l)})\|_2^2 \leq \frac{\|\nabla \Psi(\bar{y}^{l-1})\|_2^2}{2} + \frac{\varepsilon^2}{4R_y^2}$$

with probability at least $1 - 3\beta$. Therefore, after $l = \log_2(2R_y^2 \|\nabla \Psi(y^0)\|_2^2 / \varepsilon^2)$ of such restarts our method provides the point $\bar{y}^{l,p(l)}$ such that with probability at least $1 - 3l\beta$

$$\|\nabla \Psi(\bar{y}^{l,p(l)})\|_2^2 \leq \frac{\|\nabla \Psi(y^0)\|_2^2}{2^l} + \frac{\varepsilon^2}{4R_y^2} \sum_{k=0}^{l-1} 2^{-k} \leq \frac{\varepsilon^2}{2R_y^2} + \frac{\varepsilon^2}{4R_y^2} \cdot 2 = \frac{\varepsilon^2}{R_y^2}.$$

The approach informally described above is stated as Algorithm 11.

Algorithm 11 Restarted-RRMA-AC-SA²

Input: y^0 — starting point, l — number of restarts, $\{\hat{r}_k\}_{k=1}^l, \{\bar{r}_k\}_{k=1}^l$ — batch-sizes, $\{p_k\}_{k=1}^l$ — amplification parameters

1: Choose the smallest integer $\bar{N} > 1$ such that $\frac{CL_\Psi^2 \ln^4 \bar{N}}{\mu_\Psi^2 \bar{N}^4} \leq \frac{1}{32}$

2: $\bar{y}^{0,p(0)} \leftarrow y^0$

3: **for** $k = 1, \dots, l$ **do**

4: Compute $\nabla \Psi(\bar{y}^{k-1,p(k-1)}, \xi^{k-1,p(k-1)}, \hat{r}_k)$

5: $r_k \leftarrow \max \left\{ 1, \frac{64C\sigma_\Psi^2 \ln^6 \bar{N}}{\bar{N} \|\nabla \Psi(\bar{y}^{k-1,p(k-1)}, \xi^{k-1,p(k-1)}, \hat{r}_k)\|_2^2} \right\}$

6: Run p_k independent trajectories of RRMA-AC-SA² for \bar{N} iterations with batch-size r_k with $\bar{y}^{k-1,p(k-1)}$ as a starting point and get outputs $\bar{y}^{k,1}, \dots, \bar{y}^{k,p_k}$

7: Compute $\nabla \Psi(\bar{y}^{k,1}, \xi^{k,1}, \bar{r}_k), \dots, \nabla \Psi(\bar{y}^{k,p_k}, \xi^{k,p_k}, \bar{r}_k)$

8: $p(k) \leftarrow \operatorname{argmin}_{p=1, \dots, p_k} \|\nabla \Psi(\bar{y}^{k,p}, \xi^{k,p}, \bar{r}_k)\|_2$

9: **end for**

Output: $\bar{y}^{l,p(l)}$.

Theorem 12 (Theorem 5.6 from [51]). Assume that Ψ is μ_Ψ -strongly convex and L_Ψ -smooth. If Algorithm 11 is run with

$$\begin{aligned} l &= \max \left\{ 1, \log_2 \frac{2R_y^2 \|\nabla \Psi(y^0)\|_2^2}{\varepsilon^2} \right\} \\ \hat{r}_k &= \max \left\{ 1, \frac{4\sigma_\Psi^2 \left(1 + \sqrt{3 \ln \frac{l}{\beta}}\right)^2 R_y^2}{\varepsilon^2} \right\}, \quad r_k = \max \left\{ 1, \frac{64C\sigma_\Psi^2 \ln^6 \bar{N}}{\bar{N} \|\nabla \Psi(\bar{y}^{k-1,p(k-1)}, \xi^{k-1,p(k-1)}, \hat{r}_k)\|_2^2} \right\}, \\ p_k &= \max \left\{ 1, \log_2 \frac{l}{\beta} \right\}, \quad \bar{r}_k = \max \left\{ 1, \frac{128\sigma_\Psi^2 \left(1 + \sqrt{3 \ln \frac{lp_k}{\beta}}\right)^2 R_y^2}{\varepsilon^2} \right\} \end{aligned} \quad (53)$$

for all $k = 1, \dots, l$ where $\bar{N} > 1$ is such that $\frac{CL_\Psi^2 \ln^4 \bar{N}}{\mu_\Psi^2 \bar{N}^4} \leq \frac{1}{32}$, $\beta \in (0, 1/3)$ and $\varepsilon > 0$, then with probability at least $1 - 3\beta$

$$\|\nabla \Psi(\bar{y}^{l,p(l)})\|_2 \leq \frac{\varepsilon}{R_y} \quad (54)$$

and the total number of the oracle calls equals

$$\sum_{k=1}^l (\hat{r}_k + \bar{N}p_k r_k + p_k \bar{r}_k) = \tilde{O} \left(\max \left\{ \sqrt{\frac{L_\Psi}{\mu_\Psi}}, \frac{\sigma_\Psi^2 R_y^2}{\varepsilon^2} \right\} \right). \quad (55)$$

Corollary 2 (Corollary 5.7 from [51]). *Under assumptions of Theorem 12 we get that with probability at least $1 - 3\beta$*

$$\|\bar{y}^{l,p(l)} - y^*\|_2 \leq \frac{\varepsilon}{\mu_\Psi R_y}, \quad (56)$$

where $\beta \in (0, 1/3)$ the total number of the oracle calls is defined in (55).

Now we are ready to present convergence guarantees for the primal function and variables.

Corollary 3 (Corollary 5.8 from [51]). *Let the assumptions of Theorem 12 hold. Assume that f is L_f -Lipschitz continuous on $B_{R_f}(0)$ where*

$$R_f = \left(\frac{\mu_\Psi}{8\sqrt{\lambda_{\max}(A^\top A)}} + \frac{\sqrt{\lambda_{\max}(A^\top A)}}{\mu} + \frac{R_x}{R_y} \right) R_y$$

and $R_x = \|x(A^\top y^*)\|_2$. Then, with probability at least $1 - 4\beta$

$$f(x^l) - f(x^*) \leq \left(2 + \frac{L_f}{8R_y\sqrt{\lambda_{\max}(A^\top A)}} \right) \varepsilon, \quad \|Ax^l\|_2 \leq \frac{9\varepsilon}{8R_y}, \quad (57)$$

where $\beta \in (0, 1/4)$, $\varepsilon \in (0, \mu_\Psi R_y^2)$ $x^l \stackrel{\text{def}}{=} x(A^\top \bar{y}^{l,p(l)}, \xi^{l,p(l)}, \bar{r}_l)$ and to achieve it we need the following number of oracle calls:

$$\sum_{k=1}^l (\hat{r}_k + \bar{N}p_k r_k + p_k \bar{r}_k) = \tilde{O} \left(\max \left\{ \sqrt{\frac{L}{\mu} \chi(A^\top A)}, \frac{\sigma_x^2 M^2}{\varepsilon^2} \chi(A^\top A) \right\} \right) \quad (58)$$

where $M = \|\nabla f(x^*)\|_2$.

3.2.3 Direct Acceleration for Strongly Convex Dual Function

First of all, we consider the following minimization problem:

$$\min_{y \in \mathbb{R}^n} \psi(y), \quad (59)$$

where $\psi(y)$ is μ_Ψ -strongly convex and L_Ψ -smooth. We use the same notation to define the objective in (59) as for the dual function from (22) because later in the section we apply the algorithm introduced below to the (22), but for now it is not important that ψ is a dual function for (21) and we prefer to consider more general situation. As in Section 3.2.1, we do not assume that we have an access to the exact gradient of $\psi(y)$ and consider instead of it biased stochastic gradient $\tilde{\nabla}\psi(y, \xi)$ satisfying inequalities (39) and (40) with $\delta \geq 0$ and $\sigma_\Psi \geq 0$. In the main method of this section batched version of the stochastic gradient is used:

$$\tilde{\nabla}\Psi(y, \xi^k) = \frac{1}{r_k} \sum_{l=1}^{r_k} \tilde{\nabla}\psi(y, \xi^l), \quad (60)$$

where r_k is the batch-size that we leave unspecified for now. Note that $\tilde{\nabla}\Psi(y, \xi^k)$ satisfies inequalities (42) and (43).

We use Stochastic Similar Triangles Method which is stated in this section as Algorithm 12 to solve problem (59). To define the iterate z^{k+1} we use the following sequence of functions:

$$\begin{aligned} \tilde{g}_0(z) &\stackrel{\text{def}}{=} \frac{1}{2} \|z - z^0\|_2^2 + \alpha_0 \left(\psi(y^0) + \langle \tilde{\nabla}\Psi(y^0, \xi^0), z - y^0 \rangle + \frac{\mu_\Psi}{2} \|z - y^0\|_2^2 \right), \\ \tilde{g}_{k+1}(z) &\stackrel{\text{def}}{=} \tilde{g}_k(z) + \alpha_{k+1} \left(\psi(\bar{y}^{k+1}) + \langle \tilde{\nabla}\Psi(\bar{y}^{k+1}, \xi^{k+1}), z - \bar{y}^{k+1} \rangle + \frac{\mu_\Psi}{2} \|z - \bar{y}^{k+1}\|_2^2 \right) \\ &= \frac{1}{2} \|z - z^0\|_2^2 + \sum_{l=0}^{k+1} \alpha_l \left(\psi(\bar{y}^l) + \langle \tilde{\nabla}\Psi(\bar{y}^l, \xi^l), z - \bar{y}^l \rangle + \frac{\mu_\Psi}{2} \|z - \bar{y}^l\|_2^2 \right) \end{aligned} \quad (61)$$

We notice that $\tilde{g}_k(z)$ is $(1 + A_k \mu_\Psi)$ -strongly convex.

Algorithm 12 Stochastic Similar Triangles Methods for strongly convex problems (SSTM_{SC})

Input: $\tilde{y}^0 = z^0 = y^0$ — starting point, N — number of iterations
1: Set $\alpha_0 = A_0 = 1/L_\Psi$
2: Get $\tilde{\nabla}\Psi(y^0, \xi^0)$ to define $\tilde{g}_0(z)$
3: **for** $k = 0, 1, \dots, N-1$ **do**
4: Choose α_{k+1} such that $A_{k+1} = A_k + \alpha_{k+1}$, $A_{k+1}(1 + A_k\mu_\Psi) = \alpha_{k+1}^2 L_\Psi$
5: $\tilde{y}^{k+1} = (A_k y^k + \alpha_{k+1} \xi^k) / A_{k+1}$
6: $z^{k+1} = \operatorname{argmin}_{z \in \mathbb{R}^n} \tilde{g}_{k+1}(z)$, where $\tilde{g}_{k+1}(z)$ is defined in (61)
7: $y^{k+1} = (A_k y^k + \alpha_{k+1} z^{k+1}) / A_{k+1}$
8: **end for**
Output: x^N

For this algorithm we have the following convergence result.

Theorem 13 (Theorem 5.11 from [51]). Assume that the function ψ is μ_Ψ -strongly convex and L_Ψ -smooth,

$$r_k = \Theta \left(\max \left\{ 1, \left(\frac{\mu_\Psi}{L_\Psi} \right)^{3/2} \frac{N^2 \sigma_\Psi^2 \ln \frac{N}{\beta}}{\varepsilon} \right\} \right),$$

i.e. $r_k \geq \frac{1}{C} \max \left\{ 1, \left(\frac{\mu_\Psi}{L_\Psi} \right)^{3/2} \frac{N^2 \sigma_\Psi^2 (1 + \sqrt{3 \ln \frac{N}{\beta}})^2}{\varepsilon} \right\}$ with positive constants $C > 0$, $\varepsilon > 0$ and $N \geq 1$. If additionally $\delta \leq \frac{GR_0}{N\sqrt{A_N}}$ and $\varepsilon \leq \frac{HR_0^2}{A_N}$ where $R_0 = \|y^* - y^0\|_2$ and Algorithm 12 is run for N iterations, then with probability at least $1 - 3\beta$

$$\|y^N - y^*\|_2^2 \leq \frac{\hat{f}^2 R_0^2}{A_N}, \quad (62)$$

where $\beta \in (0, 1/3)$,

$$\hat{g}(N) = \frac{\ln \left(\frac{N}{\beta} \right) + \ln \ln \left(\frac{\hat{b}}{b} \right)}{\left(1 + \sqrt{3 \ln \left(\frac{N}{\beta} \right)} \right)^2}, \quad b = \frac{2\sigma_1^2 \alpha_1^2 R_0^2}{r_1}, \quad D = 1 + \frac{\mu_\Psi}{L_\Psi} + \sqrt{1 + \frac{\mu_\Psi}{L_\Psi}},$$

$$\hat{B} = 8HC \left(\frac{L_\Psi}{\mu_\Psi} \right)^{3/2} DR_0^4 \left(N \left(\frac{3}{2} \right)^N + 1 \right) \left(\hat{A} + 2Dh^2G^2 + 2C \left(\frac{L_\Psi}{\mu_\Psi} \right)^{3/2} (c + 2Du^2)H \right),$$

$$h = u = \frac{2}{\mu_\Psi}, \quad c = \frac{2}{\mu_\Psi^2},$$

$$\hat{A} = \frac{1}{\mu_\Psi} + \frac{2G}{L_\Psi \mu_\Psi N \sqrt{A_N}} + \frac{2G^2}{\mu_\Psi^2 N^2} + \left(\frac{L_\Psi}{\mu_\Psi} \right)^{3/4} \frac{2\sqrt{2CH}}{L_\Psi \mu_\Psi N \sqrt{A_N}} + \left(\frac{L_\Psi}{\mu_\Psi} \right)^{3/2} \frac{4CH}{L_\Psi \mu_\Psi^2 N^2 A_N},$$

$$\hat{f} = \max \left\{ \sqrt{\frac{1}{L_\Psi}}, \frac{3\hat{B}_1 D + \sqrt{9\hat{B}_1^2 D^2 + 4\hat{A} + 8cHC \left(\frac{L_\Psi}{\mu_\Psi} \right)^{3/2}}}{2} \right\}, \quad \hat{B}_1 = hG + uC_1 \sqrt{2HC \left(\frac{L_\Psi}{\mu_\Psi} \right)^{3/2} \hat{g}(N)}$$

and C_1 is some positive constant. In other words, to achieve $\|y^N - y^*\|_2^2 \leq \varepsilon$ with probability at least $1 - 3\beta$ Algorithm 12 needs $N = \tilde{O} \left(\sqrt{\frac{L_\Psi}{\mu_\Psi}} \right)$ iterations and $\tilde{O} \left(\max \left\{ \sqrt{\frac{L_\Psi}{\mu_\Psi}}, \frac{\sigma_\Psi^2}{\varepsilon} \right\} \right)$ oracle calls where $\tilde{O}(\cdot)$ hides polylogarithmic factors depending on $L_\Psi, \mu_\Psi, R_0, \varepsilon$ and β .

Next, we apply the SSTM_{SC} to the problem (22) when the objective of the primal problem (21) is L -smooth, μ -strongly convex and L_f -Lipschitz continuous on some ball which will be specified next, i.e., we consider the same setup as in Section 3.2.1 but we additionally assume that the primal functional f has L -Lipschitz continuous gradient. As in

Section 3.2.1 we also consider the case when the gradient of the dual functional is known only through biased stochastic estimators, see (36)–(43) and the paragraphs containing these formulas.

In Section 3.2.1 and 3.2.2 we mentioned that in the considered case dual function ψ is L_ψ -smooth on \mathbb{R}^n and μ_ψ -strongly convex on $y^0 + (\text{Ker}A^\top)^\perp$ where $L_\psi = \lambda_{\max}(A^\top A)/\mu$ and $\mu_\psi = \lambda_{\min}^+(A^\top A)/L$. Using the same technique as in the proof of Theorem 11 we show next that w.l.o.g. one can assume that ψ is μ_ψ -strongly convex on \mathbb{R}^n since $\tilde{\nabla}\Psi(y, \xi^k)$ lies in $\text{Im}A = (\text{Ker}A^\top)^\perp$ by definition of $\tilde{\nabla}\Psi(y, \xi^k)$. For this purposes we need the explicit formula for z^{k+1} which follows from the equation $\nabla\tilde{g}_{k+1}(z^{k+1}) = 0$:

$$z^{k+1} = \frac{z^0}{1 + A_{k+1}\mu_\psi} + \sum_{l=0}^{k+1} \frac{\alpha_l \mu_\psi}{1 + A_{k+1}\mu_\psi} \tilde{y}^l - \frac{1}{1 + A_{k+1}\mu_\psi} \sum_{l=0}^{k+1} \alpha_l \tilde{\nabla}\Psi(\tilde{y}^l, \xi^l). \quad (63)$$

Theorem 14 (Theorem 5.12 from [51]). *For all $k \geq 0$ we have that the iterates of Algorithm 12 \tilde{y}^k, z^k, y^k lie in $y^0 + (\text{Ker}(A^\top))^\perp$.*

This theorem makes it possible to apply the result from Theorem 13 for SSTM_{SC} which is run on the problem (22).

Corollary 4 (Corollary 5.13 from [51]). *Under assumptions of Theorem 13 we get that after $N = \tilde{O}\left(\sqrt{\frac{L_\psi}{\mu_\psi}} \ln \frac{1}{\varepsilon}\right)$ iterations of Algorithm 12 which is run on the problem (22) with probability at least $1 - 3\beta$*

$$\|\nabla\psi(y^N)\|_2 \leq \frac{\varepsilon}{R_y}, \quad (64)$$

where $\beta \in (0, 1/3)$ and the total number of oracles calls equals

$$\tilde{O}\left(\max\left\{\sqrt{\frac{L_\psi}{\mu_\psi}}, \frac{\sigma_\psi^2 R_y^2}{\varepsilon^2}\right\}\right). \quad (65)$$

If additionally $\varepsilon \leq \mu_\psi R_y^2$, then with probability at least $1 - 3\beta$

$$\|y^N - y^*\|_2 \leq \frac{\varepsilon}{\mu_\psi R_y}, \quad (66)$$

$$\|y^N\|_2 \leq 2R_y \quad (67)$$

Corollary 5 (Corollary 5.14 from [51]). *Let the assumptions of Theorem 13 hold. Assume that f is L_f -Lipschitz continuous on $B_{R_f}(0)$ where*

$$R_f = \left(\sqrt{\frac{2C}{\lambda_{\max}(A^\top A)}} + G_1 + \frac{\sqrt{\lambda_{\max}(A^\top A)}}{\mu}\right) \frac{\varepsilon}{R_y} + R_x,$$

$R_x = \|x(A^\top y^*)\|_2$, $\varepsilon \leq \mu_\psi R_y^2$ and $\delta_y \leq \frac{G_1 \varepsilon}{NR_y}$ for some positive constant G_1 . Assume additionally that the last batch-size r_N is slightly bigger than other batch-sizes, i.e.

$$r_N \geq \frac{1}{C} \max\left\{1, \left(\frac{\mu_\psi}{L_\psi}\right)^{3/2} \frac{N^2 \sigma_\psi^2 \left(1 + \sqrt{3 \ln \frac{N}{\beta}}\right)^2 R_y^2}{\varepsilon^2}, \frac{\sigma_\psi^2 \left(1 + \sqrt{3 \ln \frac{N}{\beta}}\right)^2 R_y^2}{\varepsilon^2}\right\}. \quad (68)$$

Then, with probability at least $1 - 4\beta$

$$f(\tilde{x}^N) - f(x^*) \leq \left(2 + \left(\sqrt{\frac{2C}{\lambda_{\max}(A^\top A)}} + G_1\right) \frac{L_f}{R_y}\right) \varepsilon, \quad (69)$$

$$\|A\tilde{x}^N\|_2 \leq \left(1 + \sqrt{2C} + G_1 \sqrt{\lambda_{\max}(A^\top A)}\right) \frac{\varepsilon}{R_y}, \quad (70)$$

where $\beta \in (0, 1/4)$, $\tilde{x}^N \stackrel{\text{def}}{=} \tilde{x}(A^\top y^N, \xi^N, r_N)$ and to achieve it we need the total number of oracle calls including the cost of computing \tilde{x}^N equals

$$\tilde{O} \left(\max \left\{ \sqrt{\frac{L}{\mu}} \chi(A^\top A), \frac{\sigma_x^2 M^2}{\varepsilon^2} \chi(A^\top A) \right\} \right) \quad (71)$$

where $M = \|\nabla f(x^*)\|_2$.

3.3 Applications to Decentralized Distributed Optimization

In this section, we apply our results to the decentralized optimization problems. First of all, we want to add additional motivation to the problem we are focusing on. As it was stated in the introductory part of this work, we are interested in the convex optimization problem

$$\min_{x \in Q \subseteq \mathbb{R}^n} f(x), \quad (72)$$

where f is a convex function and Q is closed and convex subset of \mathbb{R}^n . More precisely, we study particular case of (72) when the objective function f could be represented as a mathematical expectation

$$f(x) = \mathbb{E}_\xi [f(x, \xi)], \quad (73)$$

where ξ is a random variable. Typically x represents the feature vector defining the model, only samples of ξ are available and the distribution of ξ is unknown. One possible way to minimize generalization error (73) is to solve empirical risk minimization or finite-sum minimization problem instead, i.e., solve (72) with the objective

$$\hat{f}(x) = \frac{1}{m} \sum_{i=1}^m f(x, \xi_i), \quad (74)$$

where m should be sufficiently large to approximate the initial problem. Indeed, if $f(x, \xi)$ is convex and M -Lipschitz continuous for all ξ , Q has finite diameter D and $\hat{x} = \operatorname{argmin}_{x \in Q} \hat{f}(x)$, then (see [23, 139]) with probability at least $1 - \beta$

$$f(\hat{x}) - \min_{x \in Q} f(x) = O \left(\sqrt{\frac{M^2 D^2 n \ln(m) \ln(n/\beta)}{m}} \right), \quad (75)$$

and if additionally $f(x, \xi)$ is μ -strongly convex for all ξ , then (see [42]) with probability at least $1 - \beta$

$$f(\hat{x}) - \min_{x \in Q} f(x) = O \left(\frac{M^2 D^2 \ln(m) \ln(m/\beta)}{\mu m} + \sqrt{\frac{M^2 D^2 \ln(1/\beta)}{m}} \right). \quad (76)$$

In other words, to solve (72)+(73) with ε functional accuracy via minimization of empirical risk (74) it is needed to have $m = \tilde{O}(M^2 D^2 n / \varepsilon^2)$ in the convex case and $m = \tilde{O}(\max\{M^2 D^2 / \mu \varepsilon, M^2 D^2 / \varepsilon^2\})$ in the μ -strongly convex case where $\tilde{O}(\cdot)$ hides a constant factor, a logarithmic factor of $1/\beta$ and a polylogarithmic factor of $1/\varepsilon$.

Stochastic first-order methods such as Stochastic Gradient Descent (SGD) [57, 114, 119, 127, 160] or its accelerated variants like AC-SA [83] or Similar Triangles Method (STM) [39, 47, 117] are very popular choice to solve either (72)+(73) or (72)+(74). In contrast with their cheap iterations in terms of computational cost, these methods converge only to the neighbourhood of the solution, i.e., to the ball centered at the optimality and radius proportional to the standard deviation of the stochastic estimator. For the particular case of finite-sum minimization problem one can solve this issue via variance-reduction trick [26, 54, 64, 137] and its accelerated variants [5, 173, 174]. Unfortunately, this technique is not applicable in general for the problems of type (72)+(73). Another possible way to reduce the variance is mini-batching. When the objective function is L -smooth one can accelerate the computations of batches using parallelization [27, 38, 47, 49], and it is one of the examples where centralized distributed optimization appears naturally [13].

In other words, in some situations, e.g., when the number of samples m is too big, it is preferable in practice to split the data into q blocks, assign each block to the separate worker, e.g., processor, and organize computation of the gradient or stochastic gradient in the parallel or distributed manner. Moreover, in view of (75)-(76) sometimes to solve an expectation minimization problem it is needed to have such a big number of samples that corresponding information (e.g. some objects like images, videos and etc.) cannot be stored on 1 machine because of the memory limitations (see Section 3.5 for the detailed example of such a situation). Then, we can rewrite the objective function in the following form

$$f(x) = \frac{1}{q} \sum_{i=1}^q f_i(x), \quad f_i(x) = \mathbb{E}_{\xi_i} [f(x, \xi_i)] \text{ or } f_i(x) = \frac{1}{s_i} \sum_{j=1}^{s_i} f(x, \xi_{ij}). \quad (77)$$

Here f_i corresponds to the loss on the i -th data block and could be also represented as an expectation or a finite sum. So, the general idea for parallel optimization is to compute gradients or stochastic gradients by each worker, then aggregate the results by the master node and broadcast new iterate or needed information to obtain the new iterate back to the workers.

The visual simplicity of the parallel scheme hides synchronization drawback and high requirement to master node [134]. The big line of works is aimed to solve this issue via periodical synchronization [72, 147, 171, 165, 164, 75, 53], error-compensation [71, 148, 16, 55], quantization [4, 61, 62, 108, 163] or combination of these techniques [11, 105].

However, in this work we mainly focus on another approach to deal with aforementioned drawbacks — decentralized distributed optimization [13, 73]. It is based on two basic principles: every node communicates only with its neighbours and communications are performed simultaneously. Moreover, this architecture is more robust, e.g., it can be applied to time-varying (wireless) communication networks [133].

But let us consider first the centralized or parallel architecture. As we mentioned in the introduction, when the objective function is L -smooth one can compute batches in parallel [27, 38, 47, 49] in order to accelerate the work of the method and get the method (see Section 3 from [51] for the details) using

$$O\left(\frac{\sigma^2 R^2 / \varepsilon^2}{\sqrt{LR^2 / \varepsilon}}\right) \text{ or } O\left(\frac{\sigma^2 / \mu \varepsilon}{\sqrt{L/\mu} \ln(\mu R^2 / \varepsilon)}\right) \quad (78)$$

workers and having the working time proportional to the number of iterations of an accelerated first-order method. However, the number of workers defined in (78) could be too big in order to use such an approach in practice. But still computing the batches in parallel even with much smaller number of workers could reduce the working time of the method if the communication is fast enough.

Besides the computation of batches in parallel for the general type of problem (72)+(73), parallel optimization is often applied to the finite-sum minimization problems (72)+(74) or (72)+(77) that we rewrite here in the following form:

$$\min_{x \in Q \subseteq \mathbb{R}^n} f(x) = \frac{1}{m} \sum_{k=1}^m f_k(x). \quad (79)$$

We notice that in this section m is a number of workers and $f_k(x)$ is known only for the k -th worker. Consider the situation when workers are connected in a network and one can construct a spanning tree for this network. Assume that the diameter of the obtained graph equals d , i.e., the height of the tree — maximal distance (in terms of connections) between the root and a leaf [134]. If we run Similar Triangles Methods (STM, [47]) on such a spanning tree then we will get that the number of communication rounds will be

$$O\left(dN + d \min\left\{\frac{\sigma^2 R^2}{\varepsilon^2} \ln\left(\frac{\sqrt{LR^2 / \varepsilon}}{\beta}\right), \frac{\sigma^2}{\mu \varepsilon} \ln\left(\frac{LR^2}{\varepsilon}\right) \ln\left(\frac{\sqrt{L/\mu}}{\beta}\right)\right\}\right),$$

where

$$N = O\left(\min\left\{\sqrt{\frac{LR^2}{\varepsilon}}, \sqrt{\frac{L}{\mu}} \ln\left(\frac{LR^2}{\varepsilon}\right)\right\}\right).$$

Now let us consider the decentralized case when workers can communicate only with their neighbours. Next, we describe the method of how to reflect this restriction in the problem (79). Consider the Laplacian matrix $\bar{W} \in \mathbb{R}^{m \times m}$ of the network with vertices V and edges E which is defined as follows:

$$\bar{W}_{ij} = \begin{cases} -1, & \text{if } (i, j) \in E, \\ \deg(i), & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases} \quad (80)$$

where $\deg(i)$ is degree of i -th node, i.e. number of neighbours of the i -th worker. Since we consider only connected networks the matrix \bar{W} has unique eigenvector $1_m \stackrel{\text{def}}{=} (1, \dots, 1)^\top \in \mathbb{R}^m$ corresponding to the eigenvalue 0. It implies that for all vectors $a = (a_1, \dots, a_m)^\top \in \mathbb{R}^m$ the following equivalence holds:

$$a_1 = \dots = a_m \iff \bar{W}a = 0. \quad (81)$$

Now let us think about a_i as a number that i -th node stores. Then, using (81) we can use Laplacian matrix to express in the short matrix form the fact that all nodes of the network store the same number. In order to generalize it for the case when a_i are vectors from \mathbb{R}^n we should consider the matrix $W \stackrel{\text{def}}{=} \overline{W} \otimes I_n$ where \otimes represents the Kronecker product. Indeed, if we consider vectors $x_1, \dots, x_m \in \mathbb{R}^n$ and $\mathbf{x} = (x_1^\top, \dots, x_m^\top) \in \mathbb{R}^{nm}$, then (81) implies

$$x_1 = \dots = x_m \iff W\mathbf{x} = 0. \quad (82)$$

For simplicity, we also call W as a Laplacian matrix and it does not lead to misunderstanding since everywhere below we use W instead of \overline{W} . The key observation here that computation of Wx requires one round of communications when the k -th worker sends x_k to all its neighbours and receives x_j for all j such that $(k, j) \in E$, i.e. k -th worker gets vectors from all its neighbours. Note, that W is symmetric and positive semidefinite [134] and, as a consequence, \sqrt{W} exists. Moreover, we can replace W by \sqrt{W} in (82) and get the equivalent statement:

$$x_1 = \dots = x_m \iff \sqrt{W}\mathbf{x} = 0. \quad (83)$$

Using this we can rewrite the problem (79) in the following way:

$$\min_{\substack{\sqrt{W}\mathbf{x}=0, \\ x_1, \dots, x_m \in Q \subseteq \mathbb{R}^n}} f(\mathbf{x}) = \frac{1}{m} \sum_{k=1}^m f_k(x_k). \quad (84)$$

We are interested in the general case when $f_k(x_k) = \mathbb{E}_{\xi_k} [f_k(x_k, \xi_k)]$ where $\{\xi_k\}_{k=1}^m$ are independent. This type of objective can be considered as a special case of (77). Then, as it was mentioned in the introduction it is natural to use stochastic gradients $\nabla f_k(x_k, \xi_k)$ that satisfy

$$\|\mathbb{E}_{\xi_k} [\nabla f_k(x_k, \xi_k)] - \nabla f_k(x_k)\|_2 \leq \delta, \quad (85)$$

$$\mathbb{E}_{\xi_k} \left[\exp \left(\frac{\|\nabla f_k(x_k, \xi_k) - \mathbb{E}_{\xi_k} [\nabla f_k(x_k, \xi_k)]\|_2^2}{\sigma^2} \right) \right] \leq \exp(1). \quad (86)$$

Then, the stochastic gradient

$$\nabla f(\mathbf{x}, \xi) \stackrel{\text{def}}{=} \nabla f(\mathbf{x}, \{\xi_k\}_{k=1}^m) \stackrel{\text{def}}{=} \frac{1}{m} \sum_{k=1}^m \nabla f_k(x_k, \xi_k)$$

satisfies (see also (43))

$$\mathbb{E}_{\xi} \left[\exp \left(\frac{\|\nabla f(\mathbf{x}, \xi) - \mathbb{E}_{\xi} [\nabla f(\mathbf{x}, \xi)]\|_2^2}{\sigma_f^2} \right) \right] \leq \exp(1)$$

with $\sigma_f^2 = O(\sigma^2/m)$.

As always, we start with the smooth case with $Q = \mathbb{R}^n$ and assume that each f_k is L -smooth, μ -strongly convex and satisfies $\|\nabla_k f_k(x_k)\|_2 \leq M$ on some ball $B_{R_M}(x^*)$ where we use $\nabla_k f(x_k)$ to emphasize that f_k depends only on the k -th n -dimensional block of \mathbf{x} . Since the functional $f(\mathbf{x})$ in (84) has separable structure, it implies that f is L/m -smooth, μ/m -strongly convex and satisfies $\|\nabla f(\mathbf{x})\|_2 \leq M/\sqrt{m}$ on $B_{\sqrt{m}R_M}(\mathbf{x}^*)$. Indeed, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

$$\|\mathbf{x} - \mathbf{y}\|_2^2 = \sum_{k=1}^m \|x_k - y_k\|_2^2,$$

$$\|\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\|_2 = \sqrt{\frac{1}{m^2} \sum_{k=1}^m \|\nabla_k f_k(x_k) - \nabla_k f_k(y_k)\|_2^2} \leq \sqrt{\frac{L^2}{m^2} \sum_{k=1}^m \|x_k - y_k\|_2^2} = \frac{L}{m} \|\mathbf{x} - \mathbf{y}\|_2,$$

$$f(\mathbf{x}) = \frac{1}{m} \sum_{k=1}^m f_k(x_k) \geq \frac{1}{m} \sum_{k=1}^m \left(f(y_k) + \langle \nabla_k f_k(y_k), x_k - y_k \rangle + \frac{\mu}{2} \|x_k - y_k\|_2^2 \right)$$

$$= f(\mathbf{y}) + \langle \nabla f(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle + \frac{\mu}{2m} \|\mathbf{x} - \mathbf{y}\|_2^2,$$

$$\|\nabla f(\mathbf{x})\|_2^2 = \frac{1}{m^2} \sum_{k=1}^m \|\nabla_k f_k(x_k)\|_2^2.$$

Therefore, one can consider the problem (84) as (21) with $A = \sqrt{W}$ and $Q = \mathbb{R}^{nm}$. Next, if the starting point \mathbf{x}^0 is such that $\mathbf{x}^0 = ((x^0)^T, \dots, (x^0)^T)^T$ then

$$\mathbf{R}^2 \stackrel{\text{def}}{=} \|\mathbf{x}^0 - \mathbf{x}^*\|_2^2 = m\|x^0 - x^*\|_2^2 = mR^2, \quad R_y^2 \stackrel{\text{def}}{=} \|\mathbf{y}^*\|_2^2 \leq \frac{\|\nabla f(\mathbf{x}^*)\|_2^2}{\lambda_{\min}^+(W)} \leq \frac{M^2}{m\lambda_{\min}^+(W)}.$$

Now it should become clear why in Section 3.1 we paid most of our attention on number of $A^\top \mathbf{A} \mathbf{x}$ calculations. In this particular scenario $A^\top \mathbf{A} \mathbf{x} = \sqrt{W}^\top \sqrt{W} x = Wx$ which can be computed via one round of communications of each node with its neighbours as it was mentioned earlier in this section. That is, for the primal approach we can simply use the results discussed in Section 3.1. For convenience, we summarize them in Tables 3 and 4 which are obtained via plugging the parameters that we obtained above in the bounds from Section 3.1. Note that the results presented in this match the lower bounds obtained in [8] in terms of the number of communication rounds up to logarithmic factors and there is a conjecture [32] that these bounds are also optimal in terms of number of oracle calls per node for the class of methods that require optimal number of communication rounds. Recently, the very similar result about the optimal balance between number of oracle calls per node and number of communication round was proved for the case when the primal functional is convex and L -smooth and deterministic first-order oracle is available [167].

Assumptions on f_k	Method	# of communication rounds	# of $\nabla f_k(x)$ oracle calls per node
μ -strongly convex, L -smooth	D-MASG, $Q = \mathbb{R}^n$, [41]	$\tilde{O}\left(\sqrt{\frac{L}{\mu}} \chi\right)$	$\tilde{O}\left(\sqrt{\frac{L}{\mu}}\right)$
L -smooth	STP_IPS with STP as a subroutine, $Q = \mathbb{R}^n$, [51]	$\tilde{O}\left(\sqrt{\frac{LR^2}{\varepsilon}} \chi\right)$	$\tilde{O}\left(\sqrt{\frac{LR^2}{\varepsilon}}\right)$
μ -strongly convex, $\ \nabla f_k(x)\ _2 \leq M$	R-Sliding, [32, 85, 84, 87]	$\tilde{O}\left(\sqrt{\frac{M^2}{\mu\varepsilon}} \chi\right)$	$\tilde{O}\left(\frac{M^2}{\mu\varepsilon}\right)$
$\ \nabla f_k(x)\ _2 \leq M$	Sliding, [85, 84, 87]	$O\left(\sqrt{\frac{M^2 R^2}{\varepsilon^2}} \chi\right)$	$O\left(\frac{M^2 R^2}{\varepsilon^2}\right)$

Table 3 Summary of the covered results in this paper for solving (84) using primal deterministic approach from Section 3.1. First column contains assumptions on f_k , $k = 1, \dots, m$ in addition to the convexity, $\chi = \chi(W) = \lambda_{\max}(W)/\lambda_{\min}^+(W)$, where $\lambda_{\max}(W)$ and $\lambda_{\min}^+(W)$ are maximal and minimal positive eigenvalues of matrix W . All methods except D-MASG should be applied to solve (25).

Finally, consider the situation when $Q = \mathbb{R}^n$ and each f_k from (84) is dual-friendly, i.e. one can construct dual problem for (84)

$$\min_{\mathbf{y} \in \mathbb{R}^{nm}} \Psi(\mathbf{y}), \quad \text{where } \mathbf{y} = (y_1^\top, \dots, y_m^\top)^\top \in \mathbb{R}^{nm}, \quad y_1, \dots, y_m \in \mathbb{R}^n, \quad (87)$$

$$\varphi_k(y_k) = \max_{x_k \in \mathbb{R}^n} \{\langle y_k, x_k \rangle - f_k(x_k)\}, \quad (88)$$

$$\Phi(\mathbf{y}) = \frac{1}{m} \sum_{k=1}^m \varphi_k(my_k), \quad \Psi(\mathbf{y}) = \Phi(\sqrt{W}\mathbf{y}) = \frac{1}{m} \sum_{k=1}^m \varphi_k(m[\sqrt{W}\mathbf{x}]_k), \quad (89)$$

where $[\sqrt{W}\mathbf{x}]_k$ is the k -th n -dimensional block of $\sqrt{W}\mathbf{x}$. Note that

$$\begin{aligned} \max_{\mathbf{x} \in \mathbb{R}^{nm}} \{\langle \mathbf{y}, \mathbf{x} \rangle - f(\mathbf{x})\} &= \max_{\mathbf{x} \in \mathbb{R}^{nm}} \left\{ \sum_{k=1}^m \langle y_k, x_k \rangle - \frac{1}{m} \sum_{k=1}^m f_k(x_k) \right\} \\ &= \frac{1}{m} \sum_{k=1}^m \max_{x_k \in \mathbb{R}^n} \{\langle my_k, x_k \rangle - f_k(x_k)\} = \frac{1}{m} \sum_{k=1}^m \varphi_k(my_k) = \Phi(\mathbf{y}), \end{aligned}$$

so, $\Phi(\mathbf{y})$ is a dual function for $f(\mathbf{x})$. As for the primal approach, we are interested in the general case when $\varphi_k(y_k) = \mathbb{E}_{\xi_k} [\varphi_k(y_k, \xi_k)]$ where $\{\xi_k\}_{k=1}^m$ are independent and stochastic gradients $\nabla \varphi_k(x_k, \xi_k)$ satisfy

Assumptions on f_k	Method	# of communication rounds	# of $\nabla f_k(x, \xi)$ oracle calls per node
μ -strongly convex, L -smooth	D-MASG, in expectation, $Q = \mathbb{R}^n$, [41]	$\tilde{O}\left(\sqrt{\frac{L}{\mu}}\chi\right)$	$\tilde{O}\left(\max\left\{\sqrt{\frac{L}{\mu}}, \frac{\sigma^2}{\mu\varepsilon}\right\}\right)$
L -smooth	SSTP-IPS with STP as a subroutine, $Q = \mathbb{R}^n$, conjecture, [32, 51]	$\tilde{O}\left(\sqrt{\frac{LR^2}{\varepsilon}}\chi\right)$	$\tilde{O}\left(\max\left\{\sqrt{\frac{LR^2}{\varepsilon}}, \frac{\sigma^2 R^2}{\varepsilon^2}\right\}\right)$
μ -strongly convex, $\ \nabla f_k(x)\ _2 \leq M$	RS-Sliding Q is bounded, [32, 85, 84, 87]	$\tilde{O}\left(\sqrt{\frac{M^2}{\mu\varepsilon}}\chi\right)$	$\tilde{O}\left(\frac{M^2 + \sigma^2}{\mu\varepsilon}\right)$
$\ \nabla f_k(x)\ _2 \leq M$	S-Sliding Q is bounded, [85, 84, 87]	$\tilde{O}\left(\sqrt{\frac{M^2 R^2}{\varepsilon^2}}\chi\right)$	$\tilde{O}\left(\frac{(M^2 + \sigma^2)R^2}{\varepsilon^2}\right)$

Table 4 Summary of the covered results in this paper for solving (84) using primal stochastic approach from Section 3.1 with the stochastic oracle satisfying (85)-(86) with $\delta = 0$. First column contains assumptions on f_k , $k = 1, \dots, m$ in addition to the convexity, $\chi = \chi(W) = \lambda_{\max}(W)/\lambda_{\min}^+(W)$, where $\lambda_{\max}(W)$ and $\lambda_{\min}^+(W)$ are maximal and minimal positive eigenvalues of matrix W . All methods except D-MASG should be applied to solve (25). The bounds from the last two rows hold even in the case when Q is unbounded, but in the expectation (see [90]).

$$\|\mathbb{E}_{\xi_k} [\nabla \varphi_k(y_k, \xi_k)] - \nabla \varphi_k(y_k)\|_2 \leq \delta_\varphi, \quad (90)$$

$$\mathbb{E}_{\xi_k} \left[\exp \left(\frac{\|\nabla \varphi_k(y_k, \xi_k) - \mathbb{E}_{\xi_k} [\nabla \varphi_k(y_k, \xi_k)]\|_2^2}{\sigma^2} \right) \right] \leq \exp(1). \quad (91)$$

Consider the stochastic function $f_k(x_k, \xi_k)$ which is defined implicitly as follows:

$$\varphi_k(y_k, \xi_k) = \max_{x_k \in \mathbb{R}^n} \{ \langle y_k, x_k \rangle - f(x_k, \xi_k) \}. \quad (92)$$

Since

$$\nabla \Phi(\mathbf{y}) = \sum_{k=1}^m \nabla \varphi_k(m y_k) \stackrel{(35)}{=} \sum_{k=1}^m x_k(m y_k) \stackrel{\text{def}}{=} \mathbf{x}(\mathbf{y}), \quad x_k(y_k) \stackrel{\text{def}}{=} \operatorname{argmax}_{x_k \in \mathbb{R}^n} \{ \langle y_k, x_k \rangle - f_k(x_k) \}$$

it is natural to define the stochastic gradient $\nabla \Phi(\mathbf{y}, \xi)$ as follows:

$$\begin{aligned} \nabla \Phi(\mathbf{y}, \xi) &\stackrel{\text{def}}{=} \nabla \Phi(\mathbf{y}, \{\xi_k\}_{k=1}^m) \stackrel{\text{def}}{=} \sum_{k=1}^m \nabla \varphi_k(m y_k, \xi_k) \stackrel{(35)}{=} \sum_{k=1}^m x_k(m y_k, \xi_k) \stackrel{\text{def}}{=} \mathbf{x}(\mathbf{y}, \xi), \\ x_k(y_k, \xi_k) &\stackrel{\text{def}}{=} \operatorname{argmax}_{x_k \in \mathbb{R}^n} \{ \langle y_k, x_k \rangle - f_k(x_k, \xi_k) \}. \end{aligned}$$

It satisfies (see also (43))

$$\begin{aligned} \|\mathbb{E}_{\xi} [\nabla \Phi(\mathbf{y}, \xi)] - \nabla \Phi(\mathbf{y})\|_2 &\leq \delta_\Phi, \\ \mathbb{E}_{\xi} \left[\exp \left(\frac{\|\nabla \Phi(\mathbf{y}, \xi) - \mathbb{E}_{\xi} [\nabla \Phi(\mathbf{y}, \xi)]\|_2^2}{\sigma_\Phi^2} \right) \right] &\leq \exp(1) \end{aligned}$$

with $\delta_\Phi = m\delta_\varphi$ and $\sigma_\Phi^2 = O(m\sigma^2)$. Using this, we define the stochastic gradient of $\Psi(\mathbf{y})$ as $\nabla \Psi(\mathbf{y}, \xi) \stackrel{\text{def}}{=} \sqrt{W} \nabla \Phi(\sqrt{W} \mathbf{y}, \xi) = \sqrt{W} \mathbf{x}(\sqrt{W} \mathbf{y}, \xi)$ and, as a consequence, we get

$$\begin{aligned} \|\mathbb{E}_{\xi} [\nabla \Psi(\mathbf{y}, \xi)] - \nabla \Psi(\mathbf{y})\|_2 &\leq \delta_\Psi, \\ \mathbb{E}_{\xi} \left[\exp \left(\frac{\|\nabla \Psi(\mathbf{y}, \xi) - \mathbb{E}_{\xi} [\nabla \Psi(\mathbf{y}, \xi)]\|_2^2}{\sigma_\Psi^2} \right) \right] &\leq \exp(1) \end{aligned}$$

with $\delta_\Psi = \sqrt{\lambda_{\max}(W)}\delta_\Phi$ and $\sigma_\Psi = \sqrt{\lambda_{\max}(W)}\sigma_\Phi$.

Taking all of this into account we conclude that problem (87) is a special case of (22) with $A = \sqrt{W}$. To make the algorithms from Section 3.2 distributed we should change the variables in those methods via multiplying them by \sqrt{W} from the left [32, 34, 158], e.g. for the iterates of SPDSTM we will get

$$\tilde{y}^{k+1} := \sqrt{W}\tilde{y}^{k+1}, \quad z^{k+1} := \sqrt{W}z^{k+1}, \quad y^{k+1} := \sqrt{W}y^{k+1},$$

which means that it is needed to multiply lines 4-6 of Algorithm 7 by \sqrt{W} from the left. After such a change of variables all methods from Section 3.2 become suitable to run them in the distributed fashion. Besides that, it does not spoil the ability of recovering the primal variables since before the change of variables all of the methods mentioned in Section 3.2 used $\mathbf{x}(\sqrt{W}\mathbf{y})$ or $\mathbf{x}(\sqrt{W}\mathbf{y}, \xi)$ where points \mathbf{y} were some dual iterates of those methods, so, after the change of variables we should use $\mathbf{x}(\mathbf{y})$ or $\mathbf{x}(\mathbf{y}, \xi)$ respectively. Moreover, it is also possible to compute $\|\sqrt{W}\mathbf{x}\|_2^2 = \langle \mathbf{x}, W\mathbf{x} \rangle$ in the distributed fashion using consensus type algorithms: one communication step is needed to compute $W\mathbf{x}$, then each worker computes $\langle x_k, [W\mathbf{x}]_k \rangle$ locally and after that it is needed to run consensus algorithm. We summarize the results for this case in Table 5. Note that the proposed bounds are optimal in terms of the number of communication rounds up to polylogarithmic factors [8, 134, 135, 136]. Note that the lower bounds from [134, 135, 136] are presented for the convolution of two criteria: number of oracle calls per node and communication rounds. One can obtain lower bounds for the number of communication rounds itself using additional assumption that time needed for one communication is big enough and the term which corresponds to the number of oracle calls can be neglected. Regarding the number of oracle calls there is a conjecture [32] that the bounds that we present in this paper are also optimal up to polylogarithmic factors for the class of methods that require optimal number of communication rounds.

Assumptions on f_k	Method	# of communication rounds	# of $\nabla\phi_k(y, \xi)$ oracle calls per node
μ -strongly convex, L -smooth, $\ \nabla f_k(x)\ _2 \leq M$	R-RRMA-AC-SA ² (Algorithm 11), Corollary 3, SSTM _{sc} (Algorithm 12), Corollary 5	$\tilde{O}\left(\sqrt{\frac{L}{\mu}\chi}\right)$	$\tilde{O}\left(\max\left\{\sqrt{\frac{L}{\mu}\chi}, \frac{\sigma_\Phi^2 M^2}{\varepsilon^2}\chi\right\}\right)$
μ -strongly convex, $\ \nabla f_k(x)\ _2 \leq M$	SPDSTM (Algorithm 7), Theorem 8	$\tilde{O}\left(\sqrt{\frac{M^2}{\mu\varepsilon}\chi}\right)$	$\tilde{O}\left(\max\left\{\sqrt{\frac{M^2}{\mu\varepsilon}\chi}, \frac{\sigma_\Phi^2 M^2}{\varepsilon^2}\chi\right\}\right)$

Table 5 Summary of the covered results in this paper for solving (87) using dual stochastic approach from Section 3.2 with the stochastic oracle satisfying (85)-(86) with $\delta = 0$ for R-RRMA-AC-SA² and $\delta_\Phi = \tilde{O}(\varepsilon/(M\sqrt{m}\chi))$ for SSTM_{sc} and SPDSTM. First column contains assumptions on f_k , $k = 1, \dots, m$ in addition to the convexity, $\chi = \chi(W)$.

3.4 Discussion

In this section, we want to discuss some aspects of the proposed results that were not covered in the main part of this paper. First of all, we should say that in the smooth case for the primal approach our bounds for the number of communication steps coincides with the optimal bounds for the number of communication steps for parallel optimization if we substitute the diameter d of the spanning tree in the bounds for parallel optimization by $O(\sqrt{\chi(W)})$.

However, we want to discuss another interesting difference between parallel and decentralized optimization in terms of the complexity results which was noticed in [32]. From the line of works [80, 81, 82, 89] it is known that for the problem (72)+(77) (here we use m instead of q and iterator k instead of i for consistency) with L -smooth and μ -strongly convex f_k for all $k = 1, \dots, m$ the optimal number of oracle calls, i.e. calculations of of the stochastic gradients of f_k with σ^2 -subgaussian variance is

$$\tilde{O}\left(m + \sqrt{m\frac{L}{\mu} + \frac{\sigma^2}{\mu\varepsilon}}\right). \quad (93)$$

The bad news is that (93) does not work with full parallelization trick and the best possible way to parallelize it is described in [89]. However, standard accelerated scheme using mini-batched versions of the stochastic gradients without variance-reduction technique and incremental oracles which gives the bound

$$\tilde{O}\left(m\sqrt{\frac{L}{\mu}} + \frac{\sigma^2}{m\mu\varepsilon}\right) \quad (94)$$

for the number of oracle calls and it admits full parallelization. It means that in the parallel optimization setup when we have computational network with m nodes and the spanning tree for it with diameter d the number of oracle calls per node is

$$\tilde{O}\left(\sqrt{\frac{L}{\mu}} + \frac{\sigma^2}{m\mu\varepsilon}\right) = \tilde{O}\left(\max\left\{\sqrt{\frac{L}{\mu}}, \frac{\sigma^2}{m\mu\varepsilon}\right\}\right) \quad (95)$$

and the number of communication steps is

$$\tilde{O}\left(d\sqrt{\frac{L}{\mu}}\right). \quad (96)$$

However, for the decentralized setup the second row of Table 4 states that the number of communication rounds is the same as in (96) up to substitution of d by $\sqrt{\chi(W)}$ and the number of oracle calls per node is

$$\tilde{O}\left(\max\left\{\sqrt{\frac{L}{\mu}}, \frac{\sigma^2}{m\mu\varepsilon}\right\}\right) \quad (97)$$

which has m times bigger statistical term under the maximum than in (95). What is more, recently it was shown that there exists such a decentralized distributed method that requires

$$\tilde{O}\left(\frac{\sigma^2}{m\mu\varepsilon}\right)$$

stochastic gradient oracle calls per node [120, 121], but it is not optimal in terms of the number of communications. Recently a stochastic optimization method with consensus subroutine for time-varying graphs requiring $\tilde{O}(\sigma^2/(n\mu\varepsilon))$ oracle calls and $\tilde{O}(\sqrt{L/\mu}\chi)$ communications was proposed in [130]. The results of [130] can be easily extended to $\tilde{O}(\sqrt{L/\mu}\sqrt{\chi})$ communication complexity in the time-static case via employing accelerated consensus with Chebyshev acceleration. Moreover, there is a hypothesis [32] that in the smooth case the bounds from Tables 3 and 4 (rows 2 and 3) are not optimal in terms of the number of oracle calls per node and optimal ones can be found in Table 2.

3.5 Application for Population Wasserstein Barycenter Calculation

In this section we consider the problem of calculation of population Wasserstein barycenter since this example hides different interesting details connected with the theory discussed in this paper. In our presentation of this example we rely mostly on the recent works [30, 31].

3.5.1 Definitions and Properties

We define the probability simplex in \mathbb{R}^n as $S_n(1) = \{x \in \mathbb{R}_+^n \mid \sum_{i=1}^n x_i = 1\}$. One can interpret the elements of $S_n(1)$ as discrete probability measures with n shared atoms. For an arbitrary pair of measures $p, q \in S_n(1)$ we introduce the set $\Pi(p, q) = \{\pi \in \mathbb{R}_+^{n \times n} \mid \pi \mathbf{1} = p, \pi^\top \mathbf{1} = q\}$ called transportation polytope. Optimal transportation (OT) problem between measures $p, q \in S_n(1)$ is defined as follows

$$\mathcal{W}(p, q) = \min_{\pi \in \Pi(p, q)} \langle C, \pi \rangle = \min_{\pi \in \Pi(p, q)} \sum_{i, j=1}^n C_{ij} \pi_{ij} \quad (98)$$

where C is a transportation cost matrix. That is, (i, j) -th component C_{ij} of C is a cost of transportation of the unit mass from point x_i to the point x_j where points are atoms of measures from $S_n(1)$.

Next, we consider the entropic OT problem (see [122, 126])

$$\mathcal{W}_\mu(p, q) = \min_{\pi \in \Pi(p, q)} \sum_{i, j=1}^n (C_{ij} \pi_{ij} + \mu \pi_{ij} \ln \pi_{ij}). \quad (99)$$

Consider some probability measure \mathbb{P} on $S_n(1)$. Then one can define population barycenter of measures from $S_n(1)$ as

$$p_\mu^* = \operatorname{argmin}_{p \in S_n(1)} \int_{q \in S_n(1)} \mathcal{W}_\mu(p, q) d\mathbb{P}(q) = \operatorname{argmin}_{p \in S_n(1)} \underbrace{\mathbb{E}_q [\mathcal{W}_\mu(p, q)]}_{\mathcal{W}_\mu(p)}. \quad (100)$$

For a given set of samples q^1, \dots, q^m we introduce empirical barycenter as

$$\hat{p}_\mu^* = \operatorname{argmin}_{p \in S_n(1)} \underbrace{\frac{1}{m} \sum_{i=1}^m \mathcal{W}_\mu(p, q^i)}_{\hat{\mathcal{W}}(p)}. \quad (101)$$

We consider the problem (100) of finding population barycenter with some accuracy and discuss possible approaches to solve this problem in the following subsections.

However, before that, we need to mention some useful properties of $\mathcal{W}_\mu(p, q)$. First of all, one can write explicitly the dual function of $\mathcal{W}_\mu(p, q)$ for a fixed $q \in S_n(1)$ (see [25, 30]):

$$\mathcal{W}_\mu(p, q) = \max_{\lambda \in \mathbb{R}^n} \{ \langle \lambda, p \rangle - \mathcal{W}_{q, \mu}^*(\lambda) \} \quad (102)$$

$$\mathcal{W}_{q, \mu}^*(\lambda) = \mu \sum_{j=1}^n q_j \ln \left(\frac{1}{q_j} \sum_{i=1}^n \exp \left(\frac{-C_{ij} + \lambda_i}{\mu} \right) \right). \quad (103)$$

Using this representation one can deduce the following theorem.

Theorem 15 ([30]). *For an arbitrary $q \in S_n(1)$ the entropic Wasserstein distance $\mathcal{W}_\mu(\cdot, q) : S_n(1) \rightarrow \mathbb{R}$ is μ -strongly convex w.r.t. ℓ_2 -norm and M -Lipschitz continuous w.r.t. ℓ_2 -norm. Moreover, $M \leq \sqrt{n} M_\infty$ where M_∞ is Lipschitz constant of $\mathcal{W}_\mu(\cdot, q)$ w.r.t. ℓ_∞ -norm and⁵ $M_\infty = \tilde{O}(\|C\|_\infty)$.*

We also want to notice that function $\mathcal{W}_{q, \mu}^*(\lambda)$ is only strictly convex and the minimal eigenvalue of its hessian $\gamma \stackrel{\text{def}}{=} \lambda_{\min}(\nabla^2 \mathcal{W}_{q, \mu}^*(\lambda^*))$ evaluated in the solution $\lambda^* \stackrel{\text{def}}{=} \operatorname{argmax}_{\lambda \in \mathbb{R}^n} \{ \langle \lambda, p \rangle - \mathcal{W}_{q, \mu}^*(\lambda) \}$ is very small and there exist only such bounds that are exponentially small in n .

We will also use another useful relation (see [30]):

$$\nabla \mathcal{W}_\mu(p, q) = \lambda^*, \quad \langle \lambda^*, \mathbf{1} \rangle = 0 \quad (104)$$

where the gradient $\nabla \mathcal{W}_\mu(p, q)$ is taken w.r.t. the first argument.

3.5.2 SA Approach

Assume that one can obtain and use fresh samples q^1, q^2, \dots in online regime. This approach is called Stochastic Approximation (SA). It implies that at each iteration one can draw a fresh sample q^k and compute the gradient w.r.t. p of function $\mathcal{W}_\mu(p, q^k)$ which is μ -strongly convex and M -Lipschitz continuous with $M = \tilde{O}(\sqrt{n} \|C\|_\infty)$. Optimal methods for this case are based on iterations of the following form

$$p^{k+1} = \operatorname{proj}_{S_n(1)} \left(p^k - \eta_k \nabla \mathcal{W}_\mu(p^k, q^k) \right)$$

where $\operatorname{proj}_{S_n(1)}(x)$ is a projection of $x \in \mathbb{R}^n$ on $S_n(1)$ and the gradient $\nabla \mathcal{W}_\mu(p^k, q^k)$ is taken w.r.t. the first argument. One can show that restarted-SGD (R-SGD) from [67] that using biased stochastic gradients (see also [65, 46, 30]) $\tilde{\nabla} \mathcal{W}_\mu(p, q)$ such that

$$\|\tilde{\nabla} \mathcal{W}_\mu(p, q) - \nabla \mathcal{W}_\mu(p, q)\|_2 \leq \delta \quad (105)$$

⁵ Under assumption that measures are separated from zero, see the details in [21] and the proof of Proposition 2.5 from [30].

for some $\delta \geq 0$ and for all $p, q \in S_n(1)$ after N calls of this oracle produces such a point p^N that with probability at least $1 - \beta$ the following inequalities hold:

$$\mathcal{W}_\mu(p^N) - \mathcal{W}_\mu(p_\mu^*) = O\left(\frac{n\|C\|_\infty^2 \ln(N/\alpha)}{\mu N} + \delta\right) \quad (106)$$

and, as a consequence of μ -strong convexity of $\mathcal{W}_\mu(p, q)$ for all q ,

$$\|p^N - p_\mu^*\|_2 = O\left(\sqrt{\frac{n\|C\|_\infty^2 \ln(N/\alpha)}{\mu^2 N} + \frac{\delta}{\mu}}\right). \quad (107)$$

That is, to guarantee

$$\|p^N - p_\mu^*\|_2 \leq \varepsilon \quad (108)$$

with probability at least $1 - \beta$, R-SGD requires

$$\tilde{O}\left(\frac{n\|C\|_\infty^2}{\mu^2 \varepsilon^2}\right) \tilde{\nabla} \mathcal{W}_\mu(p, q) \text{ oracle calls} \quad (109)$$

under additional assumption that $\delta = O(\mu \varepsilon^2)$.

However, it is computationally hard problem to find $\nabla \mathcal{W}_\mu(p, q)$ with high-accuracy, i.e. find $\tilde{\nabla} \mathcal{W}_\mu(p, q)$ satisfying (105) with $\delta = O(\mu \varepsilon^2)$. Taking into account the relation (104) we get that it is needed to solve the problem (102) with accuracy $\delta = O(\mu \varepsilon^2)$ in terms of the distance to the optimum, i.e. it is needed to find such $\tilde{\lambda}$ that $\|\tilde{\lambda} - \lambda^*\|_2 \leq \delta$ and set $\tilde{\nabla} \mathcal{W}_\mu(p, q) = \tilde{\lambda}$. Using variants of Sinkhorn algorithm [79, 149, 58] one can show [30] that R-SGD finds point p^N such that (108) holds with probability at least $1 - \beta$ and it requires

$$\tilde{O}\left(\frac{n^3\|C\|_\infty^2}{\mu^2 \varepsilon^2} \min\left\{\exp\left(\frac{\|C\|_\infty}{\mu}\right) \left(\frac{\|C\|_\infty}{\mu} + \ln\left(\frac{\|C\|_\infty}{\gamma \mu^2 \varepsilon^4}\right)\right), \sqrt{\frac{n}{\gamma \mu^3 \varepsilon^4}}\right\}\right) \quad (110)$$

arithmetical operations.

3.5.3 SAA Approach

Now let us assume that large enough collection of samples q^1, \dots, q^m is available. Our goal is to find such $p \in S_n(1)$ that $\|\hat{p} - p_\mu^*\|_2 \leq \varepsilon$ with high probability, i.e. ε -approximation of the population barycenter, via solving empirical barycenter problem (101). This approach is called Stochastic Average Approximation (SAA). Since $\mathcal{W}_\mu(p, q^i)$ is μ -strongly convex and M -Lipschitz in p with $M = \tilde{O}(\sqrt{n}\|C\|_\infty)$ for all $i = 1, \dots, m$ we can conclude that with probability $\geq 1 - \beta$

$$\mathcal{W}_\mu(\hat{p}_\mu^*) - \mathcal{W}_\mu(p_\mu^*) \stackrel{(76)}{=} O\left(\frac{n\|C\|_\infty^2 \ln(m) \ln(m/\beta)}{\mu m} + \sqrt{\frac{n\|C\|_\infty^2 \ln(1/\beta)}{m}}\right) \quad (111)$$

where we use that the diameter of $S_n(1)$ is $O(1)$. Moreover, in [139] it was shown that one can guarantee that with probability $\geq 1 - \beta$

$$\mathcal{W}_\mu(\hat{p}_\mu^*) - \mathcal{W}_\mu(p_\mu^*) \stackrel{(76)}{=} O\left(\frac{n\|C\|_\infty^2}{\beta \mu m}\right). \quad (112)$$

Taking advantages of both inequalities we get that if

$$m = \tilde{\Omega}\left(\min\left\{\max\left\{\frac{n\|C\|_\infty^2}{\mu^2 \varepsilon^2}, \frac{n\|C\|_\infty^2}{\mu^2 \varepsilon^4}\right\}, \frac{n\|C\|_\infty^2}{\beta \mu^2 \varepsilon^2}\right\}\right) = \tilde{\Omega}\left(n \min\left\{\frac{\|C\|_\infty^2}{\mu^2 \varepsilon^4}, \frac{\|C\|_\infty^2}{\beta \mu^2 \varepsilon^2}\right\}\right) \quad (113)$$

then with probability at least $1 - \frac{\beta}{2}$

$$\|\hat{p}_\mu^* - p_\mu^*\|_2 \leq \sqrt{\frac{2}{\mu} (\mathcal{W}_\mu(\hat{p}_\mu^*) - \mathcal{W}_\mu(p_\mu^*))} \stackrel{(111), (112), (113)}{\leq} \frac{\varepsilon}{2}. \quad (114)$$

Assuming that we have such $\hat{p} \in S_n(1)$ that with probability at least $1 - \frac{\beta}{2}$ the inequality

$$\|\hat{p} - \hat{p}_\mu^*\|_2 \leq \frac{\varepsilon}{2} \quad (115)$$

holds, we apply the union bound and get that with probability $\geq 1 - \beta$

$$\|\hat{p} - p_\mu^*\|_2 \leq \|\hat{p} - \hat{p}_\mu^*\|_2 + \|\hat{p}_\mu^* - p_\mu^*\|_2 \leq \varepsilon. \quad (116)$$

It remains to describe the approach that finds such $\hat{p} \in S_n(1)$ that satisfies (116) with probability at least $1 - \beta$. Recall that in this subsection we consider the following problem

$$\hat{\mathcal{W}}_\mu(p) = \frac{1}{m} \sum_{i=1}^m \mathcal{W}_\mu(p, q^i) \rightarrow \min_{p \in S_n(1)}. \quad (117)$$

For each summand $\mathcal{W}_\mu(p, q^i)$ in the sum above we have the explicit formula (103) for the dual function $\mathcal{W}_{q^i, \mu}^*(\lambda)$. Note that one can compute the gradient of $\mathcal{W}_{q^i, \mu}^*(\lambda)$ via $O(n^2)$ arithmetical operations. What is more, $\mathcal{W}_{q^i, \mu}^*(\lambda)$ has a finite-sum structure, so, one can sample j -th component of q^i with probability q_j^i and get stochastic gradient

$$\nabla \mathcal{W}_{q^i, \mu}^*(\lambda, j) = \mu \nabla \left(\ln \left(\frac{1}{q_j^i} \sum_{i=1}^n \exp \left(\frac{-C_{ij} + \lambda_i}{\mu} \right) \right) \right) \quad (118)$$

which requires $O(n)$ arithmetical operations to be computed.

We start with the simple situation. Assume that each measures q^i are stored on m separate machines that form some network with Laplacian matrix $\bar{W} \in \mathbb{R}^{m \times m}$. For this scenario we can apply the dual approach described in Section 3.3 and apply bounds from Table 5. If for all $i = 1, \dots, m$ the i -th node computes the full gradient of dual functions $\mathcal{W}_{q^i, \mu}^*$ at each iteration then in order to find such a point \hat{p} that with probability at least $1 - \frac{\beta}{2}$

$$\hat{\mathcal{W}}_\mu(\hat{p}) - \mathcal{W}_\mu(\hat{p}_\mu^*) \leq \hat{\varepsilon}, \quad (119)$$

where $W = \bar{W} \otimes I_n$, this approach requires $\tilde{O} \left(\sqrt{\frac{n \|C\|_\infty^2}{\mu \hat{\varepsilon}}} \chi(W) \right)$ communication rounds and $\tilde{O} \left(n^{2.5} \sqrt{\frac{\|C\|_\infty^2}{\mu \hat{\varepsilon}}} \chi(W) \right)$ arithmetical operations per node to find gradients $\nabla \mathcal{W}_{q^i, \mu}^*(\lambda)$. If instead of full gradients workers use stochastic gradients $\nabla \mathcal{W}_{q^i, \mu}^*(\lambda, j)$ defined in (118) and these stochastic gradients have light-tailed distribution, i.e. satisfy the condition (91) with parameter $\sigma > 0$, then to guarantee (119) with probability $\geq 1 - \frac{\beta}{2}$ the aforementioned approach needs the same number of communications rounds and $\tilde{O} \left(n \max \left\{ \sqrt{\frac{n \|C\|_\infty^2}{\mu \hat{\varepsilon}}} \chi(W), \frac{m \sigma^2 n \|C\|_\infty^2}{\hat{\varepsilon}^2} \chi(W) \right\} \right)$ arithmetical operations per node to find gradients $\nabla \mathcal{W}_{q^i, \mu}^*(\lambda, j)$. Using μ -strong convexity of $\mathcal{W}_\mu(p, q^i)$ for all $i = 1, \dots, m$ and taking $\hat{\varepsilon} = \frac{\mu \varepsilon^2}{8}$ we get that our approach finds such a point \hat{p} that satisfies (115) with probability at least $1 - \frac{\beta}{2}$ using

$$\tilde{O} \left(\frac{\sqrt{n} \|C\|_\infty}{\mu \varepsilon} \sqrt{\chi(W)} \right) \quad \text{communication rounds} \quad (120)$$

and

$$\tilde{O} \left(n^{2.5} \frac{\|C\|_\infty}{\mu \varepsilon} \sqrt{\chi(W)} \right) \quad (121)$$

arithmetical operations per node to find gradients in the deterministic case and

$$\tilde{O} \left(n \max \left\{ \frac{\sqrt{n} \|C\|_\infty}{\mu \varepsilon} \sqrt{\chi(W)}, \frac{m \sigma^2 n \|C\|_\infty^2}{\mu^2 \varepsilon^4} \chi(W) \right\} \right)$$

arithmetical operations per node to find stochastic gradients in the stochastic case. However, the state-of-the-art theory of learning states (see (113)) that m should so large that in the stochastic case the second term in the bound for arithmetical operations typically dominates the first term and the dimensional dependence reduction from $n^{2.5}$ in the deterministic case to $n^{1.5}$ in the stochastic case is typically negligible in comparison with how much $\frac{m \sigma^2 \sqrt{n} \|C\|_\infty^2}{\mu^2 \varepsilon^4} \chi(W)$ is larger than

$\frac{\|C\|_\infty}{\mu\varepsilon} \sqrt{\chi(W)}$. That is, our theory says that it is better to use full gradients in the particular example considered in this section (see also Section 3.4). Therefore, further in the section we will assume that $\sigma^2 = 0$, i.e. workers use full gradients of dual functions $\mathcal{W}_{q^i, \mu}^*(\lambda)$.

However, bounds (120)-(121) were obtained under very restrictive at the first sight assumption that we have m workers and each worker stores only one measure which is unrealistic. One can relax this assumption in the following way. Assume that we have $\hat{l} < m$ machines connected in a network with Laplacian matrix \hat{W} and j -th machine stores $\hat{m}_j \geq 1$ measures for $j = 1, \dots, \hat{l}$ and $\sum_{j=1}^{\hat{l}} \hat{m}_j = m$. Next, for j -th machine we introduce \hat{m}_j virtual workers also connected in some network that j -th machine can emulate along with communication between virtual workers and for every virtual worker we arrange one measure, e.g. it can be implemented as an array-like data structure with some formal rules for exchanging the data between cells that emulates communications. We also assume that inside the machine we can set the preferable network for the virtual nodes in such a way that each machine emulates communication between virtual nodes and computations inside them fast enough. Let us denote the Laplacian matrix of the obtained network of m virtual nodes as \bar{W} . Then, our approach finds such a point \hat{p} that satisfies (115) with probability at least $1 - \frac{\beta}{2}$ using

$$\tilde{O} \left(\underbrace{\left(\max_{j=1, \dots, \hat{l}} T_{\text{cm}, j} \right)}_{T_{\text{cm}, \max}} \frac{\sqrt{n} \|C\|_\infty}{\mu\varepsilon} \sqrt{\chi(W)} \right) \quad (122)$$

time to perform communications and

$$\tilde{O} \left(\underbrace{\left(\max_{j=1, \dots, \hat{l}} T_{\text{cp}, j} \right)}_{T_{\text{cp}, \max}} n^{2.5} \frac{\|C\|_\infty}{\mu\varepsilon} \sqrt{\chi(W)} \right) \quad (123)$$

time for arithmetical operations per machine to find gradients where $T_{\text{cm}, j}$ is time needed for j -th machine to emulate communication between corresponding virtual nodes at each iteration and $T_{\text{cp}, j}$ is time required by j -th machine to perform 1 arithmetical operation for all corresponding virtual nodes in the gradients computation process at each iteration. For example, if we have only one machine and network of virtual nodes forms a complete graph than $\chi(W) = 1$, but $T_{\text{cm}, \max}$ and $T_{\text{cp}, \max}$ can be large and to reduce the running time one should use more powerful machine. In contrast, if we have m machines connected in a star-graph than $T_{\text{cm}, \max}$ and $T_{\text{cp}, \max}$ will be much smaller, but $\chi(W)$ will be of order m which is large. Therefore, it is very important to choose balanced architecture of the network at least for virtual nodes per machine if it is possible. This question requires a separate thorough study and lies out of scope of this paper.

3.5.4 SA vs SAA comparison

Recall that in SA approach we assume that it is possible to sample new measures in online regime which means that the computational process is performed on one machine, whereas in SAA approach we assume that large enough collection of measures is distributed among the network of machines that form some computational network. In practice measures from $S_n(1)$ correspond to some images. As one can see from the complexity bounds, both SA and SAA approaches require large number of samples to learn the population barycenter defined in (100). If these samples are images, then they typically cannot be stored in RAM of one computer. Therefore, it is natural to use distributed systems to store the data.

Now let us compare complexity bounds for SA and SAA. We summarize them in Table 6. When the communication is fast enough and μ is small we typically have that SAA approach significantly outperforms SA approach in terms of the complexity as well even for communication architectures with big $\chi(W)$. Therefore, for balanced architecture one can expect that SAA approach will outperform SA even more.

To conclude, we state that population barycenter computation is a natural example when it is typically much more preferable to use distributed algorithms with dual oracle instead of SA approach in terms of memory and complexity bounds.

Approach	Complexity
SA	$\tilde{O}\left(\frac{n^3\ C\ _\infty^2}{\mu^2\varepsilon^2} \min\left\{\exp\left(\frac{\ C\ _\infty}{\mu}\right)\left(\frac{\ C\ _\infty}{\mu} + \ln\left(\frac{\ C\ _\infty}{\gamma\mu^2\varepsilon^4}\right)\right), \sqrt{\frac{n}{\gamma\mu^3\varepsilon^4}}\right\}\right)$ arithmetical operations
SA, the 2-d term is smaller	$\tilde{O}\left(\frac{n^{3.5}\ C\ _\infty^2}{\sqrt{\gamma}\mu^{3.5}\varepsilon^4}\right)$ arithmetical operations
SAA	$\tilde{O}\left(T_{\text{cm,max}}\frac{\sqrt{n}\ C\ _\infty}{\mu\varepsilon}\sqrt{\chi(W)}\right)$ time to perform communications, $\tilde{O}\left(T_{\text{cp,max}}n^{2.5}\frac{\ C\ _\infty}{\mu\varepsilon}\sqrt{\chi(W)}\right)$ time for arithmetical operations per machine, where $m = \tilde{\Omega}\left(n \min\left\{\frac{\ C\ _\infty^2}{\mu^2\varepsilon^4}, \frac{\ C\ _\infty^2}{\beta\mu^2\varepsilon^2}\right\}\right)$
SAA, $\chi(W) = \Omega(m)$, $T_{\text{cm,max}} = O(1)$, $T_{\text{cp,max}} = O(1)$, $\sqrt{\beta} \geq \varepsilon$	$\tilde{O}\left(\frac{n\ C\ _\infty^2}{\sqrt{\beta}\mu^2\varepsilon^2}\right)$ communication rounds, $\tilde{O}\left(\frac{n^3\ C\ _\infty^2}{\sqrt{\beta}\mu^2\varepsilon^2}\right)$ arithmetical operations per machine

Table 6 Complexity bounds for SA and SAA approaches for computation of population barycenter defined in (100) with accuracy ε . The third row states the complexity bound for SA approach when the second term under the minimum in (110) is dominated by the first one, e.g. when μ is small enough. The last row corresponds to the case when $T_{\text{cm,max}} = O(1)$, $T_{\text{cp,max}} = O(1)$, $\sqrt{\beta} \geq \varepsilon$, e.g. $\beta = 0.01$ and $\varepsilon \leq 0.1$, and the communication network is star-like, which implies $\chi(W) = \Omega(m)$

4 Derivative-Free Distributed Optimization

As mentioned above in Section 3, the decentralized optimization problem can be rewritten as a problem with affine constraints:

$$\min_{\substack{\sqrt{W}\mathbf{x}=0, \\ x_1, \dots, x_m \in Q}} f(\mathbf{x}) = \frac{1}{m} \sum_{i=1}^m f_i(x_i), \quad (124)$$

where we use matrix $W \stackrel{\text{def}}{=} \bar{W} \otimes I_n$ for Laplacian matrix $\bar{W} = \|\bar{W}_{ij}\|_{i,j=1,1}^{m,m} \in \mathbb{R}^{m \times m}$ of the connection graph. In turn, the problem with affine constraints:

$$\min_{Ax=0, x \in Q} f(x),$$

is rewritten in a penalized form as follows:

$$\min_{x \in Q} F(x) = f(x) + \frac{R_y^2}{\varepsilon} \|Ax\|_2^2, \quad (125)$$

with some positive constants ε and R_y (for details see Section 3). As a result, we have a classical composite optimization problem, therefore this section will focus on this problem. In what follows, we will rely on work [15]. Note that the work [146] with a similar results has recently appeared (unlike work [15], it considers a more practical one-point feedback – for a more detailed explanation of the difference, see [146]). Note also, that results of [15, 146] can be generalized for saddle-point problems by using proper version of Sliding technique [88]. We will find out a method based on the Sliding Algorithm (see [84] and Section 3) for the convex composite optimization problem with smooth and non-smooth terms. One can find gradient-free methods for distributed optimization in the literature (see [97, 153]), but the method that will be discussed further is the first, which combines zeroth-order and first-order oracles. Its uses the first-order oracle for the smooth part and the zeroth-order oracle for the non-smooth part.

4.1 Theoretical part

4.1.1 Convex Case

We consider⁶ the composite optimization problem

⁶ The narrative in this section follows [15].

$$\min_{x \in Q} \Psi_0(x) = f(x) + g(x). \quad (126)$$

In this part of paper, we will work not in the Euclidean norm $\|\cdot\|_2$, but in a certain norm $\|\cdot\|$ (and the dual norm $\|\cdot\|_*$ for the norm $\|\cdot\|$). Also define the Bregman divergence associated with some function $v(x)$, which is 1-strongly convex w.r.t. $\|\cdot\|$ -norm and differentiable on Q , as follows

$$V(x, y) = v(y) - v(x) - \langle \nabla v(x), y - x \rangle, \quad \forall x, y \in Q.$$

The use of Bregman divergence and special norms allows taking into account the geometric setup of the problem. For example, when we work with the problem in a probability simplex, it seems natural to use the $\|\cdot\|_1$ -norm and the Kullback–Leibler divergence.

Next, we introduce some assumptions for problem (126): $Q \subseteq \mathbb{R}^n$ is a compact and convex set with diameter D_Q in $\|\cdot\|$ -norm, function g is convex and L -smooth on Q w.r.t. norm $\|\cdot\|$, i.e.

$$\|\nabla g(x) - \nabla g(y)\|_* \leq L\|x - y\|, \quad \forall x, y \in Q,$$

f is convex differentiable function on Q .

Assume that we have an access to the first-order oracle for g , i.e. gradient $\nabla g(x)$ is available, and to the biased stochastic zeroth-order oracle for f (see also [52, 19]) that for a given point x returns noisy value $\tilde{f}(x, \xi)$ such that

$$\tilde{f}(x, \xi) = f(x, \xi) + \Delta(x), \quad (127)$$

where $\Delta(x)$ is a bounded noise of unknown nature

$$|\Delta(x)| \leq \Delta$$

and random variable ξ is such that

$$\mathbb{E}[f(x, \xi)] = f(x).$$

Additionally, we assume that for all $x \in Q_s$ ($s \leq D_Q$)

$$\|\nabla f(x, \xi)\|_2 \leq M(\xi), \quad \mathbb{E}[M^2(\xi)] = M^2.$$

It is important to note that for the function $f(x)$ these assumptions are made only for theoretical estimates; we have no real access to $\nabla f(x)$. The question is how to replace the gradient of the function $f(x)$. The easiest way is to collect gradient completely using finite differences:

$$f'_{\text{full}}(x, \xi) = \frac{1}{r} \sum_{i=1}^n (\tilde{f}(x + rh_i, \xi) - \tilde{f}(x - rh_i, \xi)) h_i, \quad (128)$$

here we consider a standard orthogonal normalized basis $\{h_1, \dots, h_n\}$. This way we really get a vector close to the gradient. The obvious disadvantage of this method is that one need to call the oracle for $\tilde{f}(x, \xi)$ $2n$ times. Another way is to use random direction e uniformly distributed on the Euclidean sphere (see [118, 141]):

$$\tilde{f}'_r(x, \xi, e) = \frac{n}{2r} (\tilde{f}(x + re, \xi) - \tilde{f}(x - re, \xi))e. \quad (129)$$

In particular, the authors of [15] use this approximation.

Now another problem arises – we need to combine the zeroth-order and first-order oracles for different parts of the composite problem. It seems natural that the gradient-free oracle should be called more often than the gradient one. The authors of paper [15] solve this problem and propose to apply the algorithm based on Lan's Sliding [84]. The basic idea is that we fix ∇g and iterate through the inner loop (PS procedure), changing only the point x in $\tilde{f}'_r(x, \xi, e)$.

Algorithm 13 Zeroth-Order Sliding Algorithm (zOSA)**Input:** Initial point $x_0 \in Q$ and iteration limit N .Let $\beta_k \in \mathcal{R}_{++}$, $\gamma_k \in \mathcal{R}_+$, and $T_k \in \mathbb{N}$, $k = 1, 2, \dots$, be given and set $\bar{x}_0 = x_0$.**for** $k = 1, 2, \dots, N$ **do**1. Set $\underline{x}_k = (1 - \gamma_k)\bar{x}_{k-1} + \gamma_k x_{k-1}$, and let $h_k(\cdot) \equiv l_g(\underline{x}_k, \cdot)$ be defined in (130).

2. Set

$$(x_k, \tilde{x}_k) = \text{PS}(h_k, x_{k-1}, \beta_k, T_k);$$

3. Set $\bar{x}_k = (1 - \gamma_k)\bar{x}_{k-1} + \gamma_k \tilde{x}_k$.**end for****Output:** \bar{x}_N .

The PS (prox-sliding) procedure.

procedure: $(x^+, \tilde{x}^+) = \text{PS}(h, x, \beta, T)$ Let the parameters $p_t \in \mathbb{R}_{++}$ and $\theta_t \in [0, 1]$, $t = 1, \dots$, be given. Set $u_0 = \tilde{u}_0 = x$.**for** $t = 1, 2, \dots, T$ **do**

$$u_t = \underset{u \in Q}{\operatorname{argmin}} \left\{ h(u) + \langle \tilde{f}'_r(u_{t-1}, \xi_{t-1}, e_{t-1}), u \rangle + \beta V(x, u) + \beta p_t V(u_{t-1}, u) \right\},$$

$$\tilde{u}_t = (1 - \theta_t)\tilde{u}_{t-1} + \theta_t u_t.$$

end forSet $x^+ = u_T$ and $\tilde{x}^+ = \tilde{u}_T$.**end procedure:**

In the Algorithm 13 we need the following function

$$l_g(x, y) = g(x) + \langle \nabla g(x), y - x \rangle. \quad (130)$$

It is important that the random variables ξ_t are independent, and also e_t is sampled independently from previous iterations.We also note that zOSA (in contrast to the basic version – Algorithm 6) takes into account the geometric setting of the problem and uses Bregman divergence $V(x, y)$ instead of the standard Euclidean distance in prox-sliding procedure.

Next, we will briefly talk about the convergence of this method (see the full version of the analysis in [15]). First of all, we note the universal technical lemmas that forms a general approach to working with gradient-free methods for non-smooth functions. But before that we introduce a new notation:

$$F(x) = \mathbb{E}_e[f(x + re)]. \quad (131)$$

 $F(x)$ is called the smoothed function of $f(x)$. It is important to note that the function $F(x)$ is not calculated by the algorithm, this object is needed only for theoretical analysis. The first lemma states some properties of $F(x)$:**Lemma 5.** Assume that differentiable function f defined on Q_s satisfy $\|\nabla f(x)\|_2 \leq M$ with some constant $M > 0$. Then $F(x)$ defined in (131) is convex, differentiable and $F(x)$ satisfies

$$\sup_{x \in Q} |F(x) - f(x)| \leq rM, \quad \nabla F(x) = \mathbb{E}_e \left[\frac{n}{r} f(x + re) e \right], \quad \|\nabla F(x)\|_* \leq \tilde{c} p_* \sqrt{n} M,$$

where \tilde{c} is some positive constant independent of n and p_* is determined by the following relation: $\sqrt[4]{\mathbb{E}[\|e\|_*^4]} \leq p_*$.In other words, $F(x)$ provides a good approximation of $f(x)$ for small enough r .**Lemma 6.** For $\tilde{f}'_r(x, \xi, e)$ defined in (129) the following inequalities hold:

$$\|\mathbb{E}[\tilde{f}'_r(x, \xi, e)] - \nabla F(x)\|_* \leq \frac{n\Delta p_*}{r}, \quad \mathbb{E}[\|\tilde{f}'_r(x, \xi, e)\|_*^2] \leq 2p_*^2 \left(cnM^2 + \frac{n^2\Delta^2}{r^2} \right),$$

where c is some positive constant independent of n .In other words, one can consider $\tilde{f}'_r(x, \xi, e)$ as a biased stochastic gradient of $F(x)$ with bounded second moment. Therefore, instead of solving (126) directly one can focus on the problem

$$\min_{x \in Q} \Psi(x) = F(x) + g(x) \quad (132)$$

with small enough r . As mentioned earlier, this approach is universal. In particular, the analysis of gradient-free methods for non-smooth saddle-point problems can be carried out in a similar way [19].

Now we will give the main facts from [15] for $z\circ SA$ algorithm itself. The following theorem states convergence guarantees:

Theorem 16. *Suppose that $\{p_t\}_{t \geq 1}$, $\{\theta_t\}_{t \geq 1}$ are*

$$p_t = \frac{t}{2}, \quad \theta_t = \frac{2(t+1)}{t(t+3)}, \quad \text{for all } t \geq 1, \quad (133)$$

N is given, $\{\beta_k\}$, $\{\gamma_k\}$, $\{T_k\}$ are

$$\beta_k = \frac{2L}{k}, \quad \gamma_k = \frac{2}{k+1}, \quad T_k = \frac{CNp_*^2 \left(nM^2 + \frac{n^2 \Delta^2}{r^2} \right) k^2}{\tilde{D}L^2} \quad (134)$$

with $\tilde{D} = 3D_{Q,V}^2/4$, $D_{Q,V} = \max\{\sqrt{2V(x,y)} \mid x,y \in Q\}$, $D_Q = \max\{\|x-y\| \mid x,y \in Q\}$, with some positive constant C . Then for all $N \geq 1$

$$\mathbb{E}[\Psi(\bar{x}_N) - \Psi(x^*)] \leq \frac{12LD_{Q,V}^2}{N(N+1)} + \frac{n\Delta D_Q p_*}{r}.$$

Finally, need to connect the result above to the initial problem (126).

Corollary 6. *Under the assumptions of Theorem 16 we have that the following inequality holds for all $N \geq 1$:*

$$\mathbb{E}[\Psi_0(\bar{x}_N) - \Psi_0(x^*)] \leq 2rM + \frac{12LD_{Q,V}^2}{N(N+1)} + \frac{n\Delta D_Q p_*}{r}. \quad (135)$$

From (135) it follows that if

$$r = \Theta\left(\frac{\varepsilon}{M}\right), \quad \Delta = O\left(\frac{\varepsilon^2}{nMD_Q \min\{p_*, 1\}}\right)$$

and $\varepsilon = O(\sqrt{n}MD_Q)$, then the number of evaluations for ∇g and \tilde{f}_r^t , respectively, required by Algorithm 13 to find an ε -solution of (126), i.e. such \bar{x}_N that $\mathbb{E}[\Psi_0(\bar{x}_N)] - \Psi_0(x^*) \leq \varepsilon$, can be bounded by

$$O\left(\sqrt{\frac{LD_{Q,V}^2}{\varepsilon}}\right) \text{ and } O\left(\sqrt{\frac{LD_{Q,V}^2}{\varepsilon} + \frac{D_{Q,V}^2 p_*^2 n M^2}{\varepsilon^2}}\right). \quad (136)$$

It is interesting to analyze the obtained results depending on p_* , and these constants are determined depending on what geometry we have defined for our problem. For example, if we consider Euclidean proximal setup, i.e. $\|\cdot\| = \|\cdot\|_2$, $V(x,y) = \frac{1}{2}\|x-y\|_2^2$, $D_{Q,V} = D_Q$. In this case we have p_* and bound (136) for the number of (127) oracle calls reduces to

$$O\left(\sqrt{\frac{LD_Q^2}{\varepsilon} + \frac{D_Q^2 n M^2}{\varepsilon^2}}\right)$$

and the number of $\nabla g(x)$ computations remains the same. It means that our result gives the same number of first-order oracle calls as in the original Gradient Sliding algorithm, while the number of the biased stochastic zeroth-order oracle calls is n times larger in the leading term than in the analogous bound from the original first-order method. In the Euclidean case our bounds reflect the classical dimension dependence for the derivative-free optimization (see [91]).

But if we work on the probability simplex in \mathbb{R}^n and the proximal setup is entropic: $V(x,y)$ is the Kullback–Leibler divergence, i.e. $V(x,y) = \sum_{i=1}^n x_i \ln \frac{x_i}{y_i}$. In this situation we have $D_{Q,V} = \sqrt{2 \log n}$, $D_Q = 2$, $p_* = O(\log(n)/n)$ [56]. Then number of $\nabla g(x)$ calculations is bounded by $O\left(\sqrt{(L \log^2 n)/\varepsilon}\right)$. As for the number of $\tilde{f}_r^t(x, \xi, e)$ computations, we get the following bound:

$$O\left(\sqrt{\frac{L \log n}{\varepsilon} + \frac{M^2 \log^2 n}{\varepsilon^2}}\right). \quad (137)$$

4.1.2 Strongly Convex Case

In this section we additionally assume that g is μ -strongly convex w.r.t. Bregman divergence $V(x, y)$ [150], i.e. for all $x, y \in Q$

$$g(x) \geq g(y) + \langle \nabla g(y), x - y \rangle + \mu V(x, y).$$

The authors of [15] use restarts technique and get Algorithm 14.

Algorithm 14 The Multi-phase Zeroth-Order Sliding Algorithm (M-zoSA)

Input: Initial point $y_0 \in Q$ and iteration limit N_0 , initial estimate ρ_0 (s.t. $\Psi(y_0) - \Psi(y^*) \leq \rho_0$)

for $i = 1, 2, \dots, I$ **do**

 Run zoSA with $x_0 = y_{i-1}$, $N = N_0$, $\{p_t\}$ and $\{\theta_t\}$ in (133), $\{\beta_k\}$ and $\{\gamma_k\}$, $\{T_k\}$ in (134) with $\tilde{D} = \rho_0/\mu 2^i$, and y_i is output.

end for

Output: y_I .

The following theorem states the main complexity results for M-zoSA.

Theorem 17. For M-zoSA with $N_0 = 2 \lceil \sqrt{5L/\mu} \rceil$ we have

$$\mathbb{E}[\Psi(y_i) - \Psi(y^*)] \leq \frac{\rho_0}{2^i} + \frac{2n\Delta D_Q p_*}{r}.$$

Using this we derive the complexity bounds for M-zoSA.

Corollary 7. For all $N \geq 1$ the iterates of M-zoSA satisfy

$$\mathbb{E}[\Psi_0(y_i) - \Psi_0(y^*)] \leq 2rM + \frac{\rho_0}{2^i} + \frac{2n\Delta D_Q p_*}{r}. \quad (138)$$

From (138) it follows that if

$$r = \Theta\left(\frac{\varepsilon}{M}\right), \quad \Delta = O\left(\frac{\varepsilon^2}{nMD_Q \min\{p_*, 1\}}\right)$$

and $\varepsilon = O(\sqrt{n}MD_Q)$, then the number of evaluations for ∇g and \tilde{f}'_r , respectively, required by Algorithm 14 to find a ε -solution of (126) can be bounded by

$$O\left(\sqrt{\frac{L}{\mu}} \log_2 \max[1, \rho_0/\varepsilon]\right), \quad O\left(\sqrt{\frac{L}{\mu}} \log_2 \max[1, \rho_0/\varepsilon] + \frac{p_*^2 n M^2}{\mu \varepsilon}\right).$$

4.1.3 From Composite Optimization to Decentralized Distributed Optimization

Finally, we get an estimate for solving the decentralized optimization problem. With the help of (124) and (125), we reduce the original decentralized problem to the penalized problem. Next, we need to define parameters of f using parameters of local functions f_i . Assume that for each f_i we have $\|\nabla f_i(x_i)\|_2 \leq M$ for all $x_i \in Q$, all f_i are convex functions, the starting point is $\mathbf{x}_0^\top = (x_0^\top, \dots, x_0^\top)^\top$ and $\mathbf{x}_*^\top = (x_*^\top, \dots, x_*^\top)^\top$ is the optimality point for (124). Then, one can show that $\|\nabla f(\mathbf{x})\|_2 \leq M/\sqrt{m}$ on the set of such \mathbf{x} that $x_1, \dots, x_m \in Q$, $D_{Q^m}^2 = mD_Q^2$, $D_{Q^m, V}^2 = mD_{Q, V}^2$ and $R_{\mathbf{y}}^2$ from (125) is $R_{\mathbf{y}}^2 \leq M^2/m\lambda_{\min}^+(W)$. And we have estimates in the Euclidean case:

$$O\left(\sqrt{\frac{\chi(W)M^2D_Q^2}{\varepsilon^2}}\right) \text{ communication rounds and } O\left(\sqrt{\frac{\chi(W)M^2D_Q^2}{\varepsilon^2} + \frac{nD_Q^2M^2}{\varepsilon^2}}\right) \text{ calculations of } \tilde{f}(x, \xi) \text{ per node.}$$

At the same time, when we work on a simplex and use the Kullback-Leibler divergence, we get estimates similar to (137):

$$O\left(\sqrt{\frac{\chi(W)M^2 \log n}{\varepsilon^2}}\right) \text{ communication rounds and } O\left(\sqrt{\frac{\chi(W)M^2 \log n}{\varepsilon^2} + \frac{M^2 \log^2 n}{\varepsilon^2}}\right) \text{ calculations of } \tilde{f}(x, \xi) \text{ per node.}$$

The bound for the communication rounds matches the lower bound from [136, 135] and one can note that under above assumptions the obtained bound for zeroth-order oracle calculations per node is optimal up to polylogarithmic factors in the class of methods with optimal number of communication rounds (see also [32, 51]). In particular, in the Euclidean case, we lose n times (which corresponds to the case if we were to restore the gradient in the way (128)), and in the case of a simplex, only in the $\log n$ times.

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